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On the stability of the zero solution of certain second order non-linear differential equations

By L. HATVANI in Szeged

Introduction

In this paper we shall study the equation

$$x''+a(t)g(x,x')x'+b(t)f(x) = 0 \tag{E}$$

under the following assumptions:

(A_1) $a(t)\in C[0,\infty)$, $a(t)\geqq 0$; $b(t)\in C'[0,\infty)$, $b(t)>0$;

(A_2) $f(u)\in C(-\infty,\infty)$, $uf(u)>0$ $(u\neq 0)$, and $\lim\limits_{|u|\to\infty} F(u)=\infty$, where

$$F(u)=\int_0^u f(x)\,dx;$$

(A_3) $g(u,v)$ is continuous and non-negative on the (u,v) plane;

(A_4) for arbitrary $t_0\geqq 0$, x_0, x_0', (E) has a unique solution $x(t)=x(t;t_0,x_0,x_0')$ in an appropriate interval $(t_0-\xi, t_0+\xi)(\xi>0)$ with $x(t_0)=x_0$ and $x'(t_0)=x_0'$.

The zero solution of (E) is said to be *stable in the sense of Liapunov* if for every $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that every solution $x(t)=x(t;t_0,x_0,x_0')$ of (E) for which $(x_0)^2+(x_0')^2\leqq\delta$, satisfies the inequality $[x(t)]^2+[x'(t)]^2\leqq\varepsilon$ for $t\geqq t_0$ also. We say that the zero solution of (E) is *globally asymptotically stable* if every solution $x(t)=x(t;t_0,x_0,x_0')$ of (E) satisfies the relations

$$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} x'(t)=0. \tag{R}$$

In these definitions it is understood that solutions starting near the origin exist on the whole interval $t_0\leqq t<\infty$.

J. S. W. WONG [1] obtained a condition sufficient for the stability of the zero solution of (E). He also raised the question of finding conditions guaranteeing the global asymptotical stability of the zero solution of (E). We shall give an answer to this question.

In Sec. 1 we prove two lemmas concerning continuation, boundedness and oscillation of the solutions. In Sec. 2 we establish a *necessary* condition for the global asymptotical stability of the zero solution of (E) and a *sufficient* condition for the same property in case $b(t)$ is bounded on $[0, \infty)$. In Sec. 3 we investigate the case $\lim_{t\to\infty} b(t) = \infty$.

I am deeply indebted to L. PINTÉR for the help he has offered to me in the preparation of this paper.

1.

Let $x(t)$ be a solution of (E) and set

$$v(t) = \frac{[x'(t)]^2}{b(t)} + 2F(x(t)). \tag{1.1}$$

It is easy to see that

$$v'(t) = \frac{[x'(t)]^2}{b(t)}\left[2a(t)g(x(t), x'(t)) + \frac{b'(t)}{b(t)}\right]. \tag{1.2}$$

The non-negative function $v(t)$ will be called the *Liapunov function belonging to the solution* $x(t)$.

For the sake of brevity, we shall use the notation

$$q_l(t) = 2la(t) + \frac{b'(t)}{b(t)},$$

where l is an arbitrary real number.

Lemma 1.1. *Suppose that*

$$\int_0^\infty [q_k(t)]_-\, dt < \infty, \tag{1.3}$$

where k denotes the infimum of $g(u, v)$ on the plane (u, v). Then

a) *every solution $x(t; t_0, x_0, x_0')$ of* (E) *exists in* $[0, \infty)$;

b) *$v(t)$ is a function of bounded variation on* $[0, \infty)$, *and consequently tends to a finite limit as $t \to \infty$.*

Proof. a) Suppose that $x(t; t_0, x_0, x_0')$ is a solution of (E) and $[t_0, T)$ is the maximum interval to the right in which the solution $x(t)$ can be continued ($t_0 < T \leqq \infty$). By (1.2) we have on $[t_0, T)$

$$v'(t) \leqq v(t)\left[-2ka(t) - \frac{b'(t)}{b(t)}\right]_+ = v(t)\,[q_k(t)]_-, \tag{1.4}$$

therefore

$$\int_{t_0}^{t} \frac{v'(s)}{v(s)}\, ds \leqq \int_{t_0} [q_k(s)]_-\, ds,$$

i.e.

(1. 5) $$v(t) \leqq v(t_0) \exp\Big(\int_{t_0}^{T} [q_k(s)]_-\, ds\Big) = C_1,$$

thus $v(t)$ is bounded, consequently the functions $x(t)$, $x'(t)$ also are bounded on every finite subinterval of $[t_0, T)$.

Suppose now that $T < \infty$. Then $x(t)$ and $x'(t)$ are bounded on $[t_0, T)$ and by virtue of (E) $x''(t)$ is bounded too on the same interval. But $x(t)$ cannot be extended to the right of T, therefore $\lim_{t \to T-0} x(t)$ and $\lim_{t \to T-0} x'(t)$ cannot both exist, and thus either $x'(t)$ or $x''(t)$ is unbounded on $[t_0, T)$. The assumption $T < \infty$ has led to a contradiction, i.e. $x(t)$ exists in $[t_0, \infty)$.

Likewise, $x(t)$ can be continued to the left of t_0.

b) (1. 4) and (1. 5) imply $[v'(t)]_+ \leqq C_1 [q_k(t)]_-$. Since $v(t) \geqq 0$, we have

$$\int_0^\infty [v'(t)]_-\, dt \leqq v(0) + \int_0^\infty [v'(t)]_+\, dt \leqq v(0) + C_2,$$

where $C_2 = C_1 \int_0^\infty [q_k(t)]_-\, dt$; hence

$$\int_0^\infty |v'(t)|\, dt = \int_0^\infty \big([v'(t)]_+ + [v'(t)]_-\big)\, dt < \infty,$$

i.e. $v(t)$ is a function of bounded variation on $[0, \infty)$.

Corollary 1. 1. *If* (1. 3) *holds, then every solution* $x(t)$ *of* (E), *and* $x'(t)[b(t)]^{-\frac{1}{2}}$ *also, are bounded.*

Proof. In view of b), assumption (A_2) and (1. 1), the statement is obvious.

Lemma 1. 2. *Every solution* $x(t)$ *of* (E) *is either oscillatory or monotonic on an appropriate interval* $[T_0, \infty)$.

Proof. The zero solution of (E) obviously satisfies the statement of the lemma. Suppose now that $x(t) \not\equiv 0$. Then, as a consequence of the uniqueness of the zero solution of (E), $x(t)$ and $x'(t)$ have only zeros of multiplicity one and these zeros form a discrete set in every finite interval. Now to prove the lemma it is sufficient to show that between any two consecutive zeros of $x'(t)$ there is one and only one zero of $x(t)$.

Let t', t'' be two consecutive zeros of $x'(t)$. By virtue of (E) we have $x(t')x''(t')<0$, $x(t'')x''(t'')<0$, and therefore $x(t)$ has successive extremal values in t', t'', thus one of them is a maximum point, the other is a minimum point of $x(t)$. Consequently, $x''(t')$ and $x''(t'')$ are of opposite signs. Hence $x(t')$ and $x(t'')$ are also of opposite signs, and therefore $x(t)$ vanishes at some point of (t', t''). If $x(t)$ vanishes at least twice on (t', t''), then $x'(t)$ also has a zero in the same interval. This contradicts the fact that t', t'' are two consecutive zeros of $x'(t)$.

2.

Theorem 2. 1. *If*

$$\liminf_{t\to\infty} b(t) > 0 \tag{2. 1}$$

and the zero solution of (E) *is globally asymptotically stable, then*

$$\int_0^\infty [q_K(t)]_+\,dt = \infty,$$

where K is an arbitrary real number greater than $g(0, 0)$.

Proof. Let $x(t)$ be an arbitrary solution of (E). The zero solution being globally asymptotically stable, it follows from (R) and (2. 1) that $v(t)$ tends to 0 as $t\to\infty$. Since $K>g(0, 0)$ and $g(u, v)$ is continuous, there exists a $\delta>0$ such that if $u^2+v^2<\delta$ then $g(u, v)<K$. Furthermore, because of (R) there exists a $T>0$ such that if $t\geqq T$ then $[x(t)]^2+[x'(t)]^2<\delta$, and hence $g\big(x(t), x'(t)\big)<K$, provided $t\geqq T$. Thus, by (1. 1), (1. 2) and assumption (A_2), we have

$$\frac{v'(t)}{v(t)} = -\frac{1}{v(t)}\frac{[x'(t)]^2}{b(t)}\left[2a(t)g\big(x(t), x'(t)\big)+\frac{b'(t)}{b(t)}\right] \geqq -\left[2Ka(t)+\frac{b'(t)}{b(t)}\right]_+$$

on $[T, \infty)$, and therefore

$$\int_T^t \frac{v'(s)}{v(s)}\,ds = \ln\frac{v(t)}{v(T)} \geqq -\int_T^t [q_K(s)]_+\,ds$$

on the same interval. Since $v(t)$ tends to 0 as $t\to\infty$,

$$\int_0^\infty [q_K(s)]_+\,ds \geqq \int_T^\infty [q_K(s)]_+\,ds = \infty$$

holds, which was to be proved.

Theorem 2. 2. *Suppose that $a(t)$ and $b(t)$ are bounded on $[0, \infty)$, furthermore* (1. 3) *and* (2. 1) *are satisfied. If*

$$\int_S [q_k(t)]_+ \, dt = \infty \tag{2. 2}$$

holds on every set $S = \bigcup_{n=1}^{\infty} (a_n, b_n)$ *such that*

$$0 \leqq a_1, \quad a_n < b_n < a_{n+1}, \quad b_n - a_n \geqq \delta > 0 \qquad (n = 1, 2, 3, \ldots),$$

then the zero solution of (E) *is globally asymptotically stable.*

Remark 2. 1. If, say $q_k(t)$ satisfies $[q_k(t)]_+ \geqq \alpha > 0$ on $[0, \infty)$, or it is non-negative, periodic and does not vanish identically on any subinterval of $[0, \infty)$, then (2. 2) is obviously satisfied.

It is easy to prove that (2. 2) and the following statement are equivalent: for every $\delta > 0$

$$\liminf_{t \to \infty} \int_t^{t+\delta} [q_k(s)]_+ \, ds > 0$$

is valid.

Proof. Let $x(t)$ be a solution of (E). By Lemma 1. 1 $x(t)$ exists in $[0, \infty)$; $x(t)$ and $u(t) = [x'(t)]^2 [b(t)]^{-1}$ are bounded and $v(t)$ is a function of bounded variation on $[0, \infty)$.

First, we shall prove that

$$\lim_{t \to \infty} u(t) = 0. \tag{2. 3}$$

Suppose (2. 3) is false, i.e.

$$\limsup_{t \to \infty} u(t) = \lambda > 0, \tag{2. 4}$$

and consider the open unbounded set

$$H = \left\{ t : t \geqq 0, \, u(t) > \frac{\lambda}{3} \right\}. \tag{2. 5}$$

As $v(t)$ is a function of bounded variation, we have

$$\infty > \int_0^{\infty} |v'(t)| \, dt \geqq \int_H u(t) \left| 2a(t) g(x(t), x'(t)) + \frac{b'(t)}{b(t)} \right| dt \geqq \frac{\lambda}{3} \int_H [q_k(t)]_+ \, dt, \tag{2. 6}$$

and therefore, as a consequence of (2. 2), H does not contain an interval of the type (T, ∞), and hence

$$\liminf_{t\to\infty} u(t) \leqq \frac{\lambda}{3}. \tag{2.7}$$

Inequalities (2. 4) and (2. 7) imply that there exists a sequence of intervals $\omega_m = (t'_m, t''_m) \subset H$ $(m=1, 2, 3, \ldots)$ such that

$$t'_m < t''_m < t'_{m+1}, \quad u(t'_m) = u(t''_m) = \frac{\lambda}{3} \qquad (m=1, 2, 3, \ldots), \tag{2.8}$$

$\lim\limits_{m\to\infty} t'_m = \infty$ and for every m there exists a $\tau_m \in \omega_m$ with

$$u(\tau_m) = \frac{2}{3}\lambda. \tag{2.9}$$

From (2. 6) and (2. 8), by assumption (2. 2), we obtain

$$\liminf_{m\to\infty} \operatorname{mes}(\omega_m) = 0. \tag{2.10}$$

Since $u' = v' - 2[F(x)]'$, (2. 8) and (2. 9) imply

$$\frac{\lambda}{3} \leqq \int_{\omega_m} |u'(t)|\, dt \leqq \int_{\omega_m} u(t) \left| 2a(t) + g(x(t), x'(t)) + \frac{b'(t)}{b(t)} \right| dt + \\ + 2 \int_{\omega_m} |x'(t)|\, |f(x(t))|\, dt \qquad (m=1, 2, 3, \ldots); \tag{2.11}$$

moreover, in view of (2. 6) and (2. 8) we have

$$\lim_{m\to\infty} \int_{\omega_m} u(t) \left| 2a(t) g(x(t), x'(t)) + \frac{b'(t)}{b(t)} \right| dt = 0. \tag{2.12}$$

By virtue of assumption (2. 1), and the boundedness of $x(t)$ and $x'(t)[b(t)]^{-\frac{1}{2}}$, we get

$$|f(x(t))| < C_1, \quad |x'(t)| < C_2 \qquad (0 \leqq t < \infty). \tag{2.13}$$

From (2. 11) we obtain, in view of (2. 10), (2. 12) and (2. 13), the inequality

$$\frac{\lambda}{3} \leqq o(1) + 2C_1 C_2 \int_{\omega_m} ds \qquad (m \to \infty),$$

which, as a consequence of (2. 10), contradicts the fact that $\lambda > 0$; consequently (2. 3) is true. Then, in view of assumption (2. 1), it follows

$$\lim_{t\to\infty} x'(t) = 0. \tag{2.14}$$

It remains to verify that $x(t)$ tends to 0 as $t\to\infty$.

By Lemma 1. 1 $v(t)$ tends to a finite limit as $t\to\infty$, therefore it follows from (2. 3) that $\lim\limits_{t\to\infty} F(x(t))$ also exists; thus taking assumption (A_2) into consideration it is easy to see that $\lim\limits_{t\to\infty} x(t) = \nu$ exists too. By Lemma 1. 2 $x(t)$ is either oscillatory or monotonic for t large enough. In the first case we have obviously $\nu=0$; thus it is sufficient to study the second case.

Suppose that $x(t)$ is monotonic for t large enough and $\nu\neq 0$. Then, by virtue of (2. 1) and (A_2), we get

$$\liminf_{t\to\infty} b(t)\,|f(x(t))| > 0. \tag{2.15}$$

Since $a(t)$ is bounded and $g(u, v)$ is continuous, in view of (2. 14) we have

$$\lim_{t\to\infty} a(t)g(x(t), x'(t))x'(t) = 0. \tag{2.16}$$

Using (2. 15) and (2. 16) we deduce from (E) that $\liminf\limits_{t\to\infty} |x''(t)| > 0$ which contradicts the fact that $x(t)$ is bounded. Thus $\nu=0$, and this concludes the proof of the theorem.

Remark 2. 2. By taking $g(x, x')\equiv 1$ and $f(x)\equiv x^{2n-1}$, Theorem 2. 2 contains as a special case a sharpened form of a theorem of J. JONES (Theorem 4 of [3]).

3.

Theorem 3.1. *Suppose that*

$$\lim_{t\to\infty} b(t) = \infty, \qquad \inf_{-\infty<u,\, v<\infty} g(u, v) = k > 0, \tag{3.1}$$

and for any positive number C

$$\sup_{|u|<C,\, -\infty<v<\infty} g(u, v) = K_C < \infty.$$

If

$$\liminf_{t\to\infty} \frac{q_k(t)}{[b(t)]^{\frac{1}{2}}} > 0 \quad \textit{and} \quad \liminf_{t\to\infty} \frac{[b(t)]^{\frac{1}{2}}}{a(t)} > 0, \tag{3.2}$$

then for every solution $x(t)$ of (E) *we have*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x'(t)}{[b(t)]^{\frac{1}{2}}} = 0.$$

Proof. This is similar to that of Theorem 2. 2.

Let $x(t)$ be a solution of (E). By virtue of (3. 2) we have $[q_k(t)]_-\equiv 0$ on an appropriate interval $[T_0, \infty)$, thus by Lemma 1. 1 $x(t)$ exists in $[0, \infty)$; furthermore

$x(t)$ and $u(t)=[x'(t)]^2[b(t)]^{-1}$ are bounded and $v(t)$ is non-negative and decreasing on $[T_0, \infty)$.

First we shall prove that

$$\lim_{t\to\infty} u(t) = 0. \tag{3.3}$$

Suppose that (3. 3) is false, i.e. (2. 4) holds, and consider the set H defined by (2. 5). In view of (3. 2) there exists a positive number c such that if T $(T \geqq T_0)$ is large enough, then

$$v(T) > \int_T^\infty |v'(t)|\,dt = \int_T^\infty u(t)\left[2a(t)g(x(t), x'(t)) + \frac{b'(t)}{b(t)}\right] dt > \tag{3.4}$$

$$> c\frac{\lambda}{3} \int_{H\cap[T,\infty)} [b(t)]^{\frac{1}{2}}\,dt;$$

hence by (3. 1) we get that $\operatorname{mes}(H) < \infty$. Thus, the present assumptions also imply (2. 7) and there exists a sequence of intervals $\omega_m \subset H$ $(m=1, 2, 3, \dots)$ with (2. 8) and (2. 9). Combining (3. 2) and (3. 4) we obtain the estimate

$$v(T) > \int_T^\infty |v'(t)|\,dt \geqq \left(\frac{\lambda}{3}\right)^{\frac{1}{2}} \int_{H\cap[T,\infty)} |x'(t)|\frac{q_k(t)}{[b(t)]^{\frac{1}{2}}}\,dt \geqq c\left(\frac{\lambda}{3}\right)^{\frac{1}{2}} \int_{H\cap[T,\infty)} |x'(t)|\,dt,$$

from which it follows that

$$\lim_{m\to\infty} \int_{\omega_m} |x'(t)|\,dt = 0. \tag{3.5}$$

From (2. 11) using (2. 12), (3. 5) and the boundedness of $x(t)$ we get the inequality

$$\frac{\lambda}{3} \leqq o(1) + 2C_1 \int_{\omega_m} |x'(t)|\,dt = o(1) \qquad (m\to\infty),$$

which contradicts the fact, that $\lambda > 0$. Thus, (3. 3) is true.

It remains to verify that $x(t)$ tends to 0 as $t\to\infty$.

Similarly as in the proof of Theorem 2. 2, we may restrict ourselves to the case where $x(t)$ is monotonic for t large enough. Denote by v the limit of $x(t)$ as $t\to\infty$ (as $x(t)$ is bounded, v is finite), and suppose $v\neq 0$. Then $\liminf_{t\to\infty} |x'(t)| = 0$, from which it follows that for the function $w(t)=x'(t)[x(t)]^{-1}$

$$\liminf_{t\to\infty} |w(t)| = 0 \tag{3.6}$$

holds too. On the other hand, (E) implies that $w(t)$ satisfies the relation

$$w'(t) = -b(t)\left[\frac{(w(t))^2}{b(t)} + \frac{a(t)}{[b(t)]^{\frac{1}{2}}}\, g\big(x(t), x'(t)\big)\frac{w(t)}{[b(t)]^{\frac{1}{2}}} + \frac{f(x(t))}{x(t)}\right] \tag{3.7}$$

for t large enough. In consequence of (3. 3) and $v \neq 0$, we have

$$\lim_{t\to\infty} \frac{w(t)}{[b(t)]^{\frac{1}{2}}} = 0,$$

thus (3. 1), (3. 2), (3. 7), (A_2) and assumption $v \neq 0$ imply that

$$\lim_{t\to\infty} |w'(t)| = \infty. \tag{3.8}$$

This contradicts (3. 6). Therefore we have $v=0$.

This concludes the proof.

Corollary. *Suppose $b(t)$ is increasing on an interval $[T, \infty)$ and $\lim_{t\to\infty} b(t) = \infty$. If there exist positive constants k, K, c, C, T_1 such that*

$$k < g(u, v) < K \quad (-\infty < u, v < \infty), \qquad c \leqq \frac{[b(t)]^{\frac{1}{2}}}{a(t)} \leqq C \quad (T \leqq T_1 \leqq t < \infty), \tag{3.9}$$

then for every solution $x(t)$ of (E) *we have*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x'(t)}{[b(t)]^{\frac{1}{2}}} = 0.$$

Proof. If $t \in [T_1, \infty)$, then

$$\frac{q_k(t)}{[b(t)]^{\frac{1}{2}}} = \frac{1}{[b(t)]^{\frac{1}{2}}}\left[2ka(t) + \frac{b'(t)}{b(t)}\right] \geqq 2k\frac{a(t)}{[b(t)]^{\frac{1}{2}}},$$

thus (3. 9) implies (3. 2); therefore the assumptions of the theorem are satisfied.

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Über die unbedingte Konvergenz der Orthogonalreihen

Von KÁROLY TANDORI in Szeged

1. In der Arbeit [5] haben wir eine notwendige und hinreichende Bedingung für die unbedingte Konvergenz der Orthogonalreihe

$$(1) \qquad \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

angegeben, wobei $\{a_n\}$ eine reelle Koeffizientenfolge und $\{\varphi_n(x)\}$ ein orthonormiertes Funktionensystem (kurz ONS) ist; in Folgendem — wenn es nichts anderes gesagt wird — nehmen wir für das Orthogonalitätsintervall immer das Intervall $(0, 1)$. Die Reihe (1) konvergiert fast überall unbedingt, wenn sie in jeder Anordnung

$$(2) \qquad \sum_{l=1}^{\infty} a_{n(l)} \varphi_{n(l)}(x)$$

fast überall konvergiert (die Menge der Divergenzpunkte kann von der Anordnung abhängen, darum folgt aus der „unbedingten Konvergenz fast überall" die „absolute Konvergenz fast überall" nicht).

Zur Abfassung des erwähnten Resultates bilden wir die Summe

$$S = \sum_{k=0}^{\infty} \left(\sum_{n=\nu(k)+1}^{\nu(k+1)} a_n^{*2} \log^2 n \right)^{\frac{1}{2}},$$

wobei $\nu(k) = 2^{2^k}$ $(k = 0, 1, \ldots)$ ist, und $\{a_n^*\}$ eine derartige Anordnung der Folge $\{a_n\}$ bezeichnet, für die $|a_1^*| \geqq \cdots \geqq |a_n^*| \geqq \cdots$ gilt. (Ist $\{a_n\} \in l^2$, d.h. gilt $\Sigma a_n^2 < \infty$, dann existiert eine solche im absoluten Betrag monoton abnehmende Anordnung; im Falle $\{a_n\} \notin l^2$ soll man $S = \infty$ setzen. Es ist möglich, daß für eine Folge $\{a_n\}$ verschiedene solche Anordnungen existieren; für verschiedene solche Anordnungen sind aber die Werte S dieselben.) Die erwähnte Bedingung lautet folgenderweise.

A. *Dafür, daß die Reihe* (1) *für jedes* ONS $\{\varphi_n(x)\}$ *fast überall unbedingt konvergiert, ist* $S < \infty$ *notwendig und hinreichend.*

Die Notwendigkeit der Bedingung $S<\infty$ ergibt sich aus der folgenden Behaptung.

B. *Ist* $S=\infty$, *dann gibt es ein ONS* $\{\Phi_n(x)\}$ *derart, daß die Reihe*

$$\sum_{n=1}^{\infty} a_n \Phi_n(x) \tag{3}$$

in gewisser Anordnung ihrer glieder fast überall divergiert.

2. Der Beweis dieser Behauptung geht mit einer direkten Konstruktion, und das entsprechende ONS ist unbeschränkt. In dieser Arbeit werden wir zeigen, daß in dieser Behauptung das ONS $\{\Phi_n(x)\}$ beschränkt angewählt werden kann. Genauer zeigen wir den folgenden Satz.

Satz I. *Es sei* $K>1$. *Ist* $S=\infty$, *dann gibt es ein* ONS $\{\Phi_n(x)\}$ *mit*

$$|\Phi_n(x)| \leqq K \qquad (0 \leqq x \leqq 1;\quad n=1,2,\ldots) \tag{4}$$

derart, daß die Reihe (3) *in gewisser Anordnung ihrer Glieder in* (0, 1) *fast überall divergiert.*

Ein Funktionensystem $\{\Phi_n(x)\}$ mit der Eigenschaft (4) nennen wir ein *K-beschränktes* System.

Da nach der Behauptung A aus $S<\infty$ folgt, daß die Reihe (1) für jedes ONS $\{\Phi_n(x)\}$ fast überall unbedingt konvergiert, erhalten wir nun unmittelbar auch den folgenden Satz.

Satz II. *Es sei* $K>1$. *Dafür, daß die Reihe* (1) *für jedes K-beschränkte* ONS $\{\varphi_n(x)\}$ *fast überall unbedingt konvergiert, ist* $S<\infty$ *notwendig und hinreichend.*

Die allen ONS-e und die K-beschränkten ONS-e verhalten sich also vom Gesichtspunkt der unbedingten Konvergenz ähnlicherweise. Für den Fall der gewöhnlichen Konvergenz wurde es auf diese Erscheinung mehrmals aufmerksam gemacht (s. [2], [4], [8]).

Aus dem Satz II und aus der Behauptung A ergibt sich z. B. folgendes. *Ist die Reihe* (1) *für jedes* $K(>1)$*-beschränkte* ONS $\{\varphi_n(x)\}$ *fast überall unbedingt konvergent, dann konvergiert die Reihe* (1) *auch für jedes ONS fast überall unbedingt.*

Es soll betont werden, daß für den Fall $K=1$ der Satz I noch nicht bewiesen ist. Für 1-beschränkte, oder m.a.W. vorzeichensartige ONS-e anstatt des Satzes I ist nur ein schwächeres Resultat bekannt [9]. Da für vorzeichensartige ONS-e konnte man bisher auch die hinreichende Bedingung $S<\infty$ nicht abschwächen, ist also die „genaue" Bedingung der unbedingten Konvergenz für die Entwicklungen nach vorzeichensartigen ONS-en noch unbekannt.

Der originallen Beweis der Behauptung B ist sehr kompliziert und langwierig. Den stärkeren Satz I werden wir im Folgenden ziemlich einfacher beweisen.

3. Der Beweis des Satzes I beruht auf einem Hilfssatz. Im Folgenden bezeichnen $A_1, A_2, \ldots$ positive, absolute Konstante und $C_1(K), C_2(K), \ldots$ nur von K abhängige positive Konstante. Eine Funktion nennen wir Treppenfunktion in einem Intervall I, wenn es eine Zerlegung von I in endlich viele Teilintervalle derart gibt, daß die Funktion in jedem Teilintervall konstant ist; eine Menge nennen wir einfach, wenn sie die Vereinigung endlichvieler Intervalle ist. Das Lebesguesche Maß einer meßbaren Menge H bezeichnen wir mit $m(H)$. Unter log verstehen wir im Folgenden den Logarithmus mit der Basis 2. Der erwähnte Hilfssatz lautet folgenderweise.

Hilfssatz I. *Es seien $K>1$, $(C_0(K)\leqq)k_1<k_2$ ganze Zahlen (die positive nur von K abhängige ganze Zahl $C_0(K)$ wird später bestimmt) und*

$$\{a_n\} \quad (n=\nu(k_1)+1, \ldots, \nu(k_2))$$

eine im absoluten Betrag monoton abnehmende Folge. Dann gibt es ein K-beschränktes ONS von Treppenfunktionen $\psi_n^*(x)$ $(n=\nu(k_1)+1, \ldots, \nu(k_2))$ und eine einfache Menge $E^*(\subseteqq(0,1))$ mit $m(E^*)\geqq C_1(K)$ derart, daß die Summe $\sum_{n=\nu(k_1)+1}^{\nu(k_2)} a_n\psi_n^*(x)$ eine Anordnung $\sum_{l=1}^{\nu(k_2)-\nu(k_1)} a_{m(l)}\psi_{m(l)}^*(x)$ besitzt, für die

$$\max_{1\leqq\lambda\leqq\nu(k_2)-\nu(k_1)} \sum_{l=1}^{\lambda} a_{m(l)}\psi_{m(l)}^*(x) \geqq \tag{5}$$

$$\geqq C_2(K)\sum_{k=k_1}^{k_2}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}} \qquad (x\in E^*)$$

erfüllt ist.

4. Für eine Koeffizientenfolge $\{a_n\}$ bilden wir die „Norm"

$$\|\{a_n\};\infty\|^* = \sup_P \sup_{\{\varphi_n\}}\left\{\int_0^1\left(\sup_{1\leqq i\leqq j}(a_i\varphi_i(x)+\cdots+a_j\varphi_j(x))^2\right)dx\right\}^{\frac{1}{2}} =$$

$$=\lim_{N\to\infty}\max_P\sup_{\{\varphi_n\}}\left\{\int_0^1\left(\max_{1\leqq i\leqq j\leqq N}(a_i\varphi_i(x)+\cdots+a_j\varphi_j(x))^2\right)dx\right\}^{\frac{1}{2}},$$

wobei $\sup_P$, bzw. $\max_P$ bedeutet, daß das Supremum, bzw. das Maximum des entsprechenden Ausdruckes für jede Anordnung der Folge $\{a_n\}$ $(n=1,2,\ldots)$, bzw. der Folge $\{a_n\}$ $(n=1,\ldots,N)$ gebildet ist, weiterhin $\sup_{\{\varphi_n\}}$ bedeutet, das das Supremum des entsprechenden Ausdruckes für jedes ONS $\{\varphi_n(x)\}$ gebildet ist.

Es sei $K \geqq 1$. Für eine Koeffizientenfolge $\{a_n\}$ können wir auch eine andere „Norm" bilden:

$$\|\{a_n\}; K\|^* = \sup_P \sup_{|\varphi_n| \leqq K} \left\{ \int_0^1 \Big(\sup_{1 \leqq i \leqq j} (a_i \varphi_i(x) + \cdots + a_j \varphi_j(x))^2 \Big) dx \right\}^{\frac{1}{2}} =$$

$$= \lim_{N \to \infty} \max_P \sup_{|\varphi_n| \leqq K} \left\{ \int_0^1 \Big(\max_{1 \leqq i \leqq j \leqq N} (a_i \varphi_i(x) + \cdots + a_j \varphi_j(x))^2 \Big) dx \right\}^{\frac{1}{2}},$$

wobei $\sup\limits_{|\varphi_n| \leqq K}$ bedeutet, daß das Supremum für jedes K-beschränkte ONS $\{\varphi_n(x)\}$ gebildet wird. Offensichtlich gilt

$$\|\{a_n\}; K\|^* \leqq \|\{a_n\}; \infty\|^*. \tag{6}$$

Für eine Koeffizintenfolge $\{a_n\}$ setzen wir weiterhin

$$S_1 = |a_1^*| + |a_2^*| + \sum_{k=0}^{\infty} \left(\sum_{n=\nu(k)+1}^{\nu(k+1)} a_n^{*2} \log^2 n \right)^{\frac{1}{2}}$$

(zur Definition von dieser Grösse siehe die Bemerkung nach der Definition von S). In der Arbeit [7] haben wir

$$A_1 S_1 \leqq \|\{a_n\}; \infty\|^* \leqq A_2 S_1 \tag{7}$$

bewiesen.

Nun beweisen wir die Ungleichung

$$\|\{a_n\}; K\|^* \geqq C_3(K) S_1 \qquad (K > 1). \tag{8}$$

Darum einführen wir einige Bezeichnungen. Für eine Funktion $f(x)$ und für ein endliches Intervall $I = (a, b) \, (\subseteqq (0, 1))$ setzen wir

$$f(I; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & (a < x < b), \\ 0 & \text{sonst;} \end{cases}$$

weiterhin für eine Menge H $(\subseteqq (0, 1))$ bezeichnet $H(I)$ diejenige Untermenge von (a, b), die aus H mit der linearen Transformation $y = (b-a)x + a$ entsteht. Aus dem Hilfssatz I folgt

Hilfssatz I'. *Es seien $K > 1$, $k_0 (\geqq C_0(K))$ eine ganze Zahl und*

$$\{a_n\} \; (n = 1, \ldots, \nu(k_0))$$

eine im absoluten Betrag monoton abnehmende Folge. Dann gibt es ein K-beschränktes ONS von Treppenfunktionen $\psi_n(x)$ $(n = 1, \ldots, \nu(k_0))$ und eine einfache Menge

$E(\subseteqq(0,1))$ mit $m(E)\geqq C_4(K)$ derart, daß die Summe $\sum_{n=1}^{\nu(k_0)} a_n\psi_n(x)$ eine Anordnung $\sum_{l=1}^{\nu(k_0)} a_{n(l)}\psi_{n(l)}(x)$ besitzt, für die

$$\max_{1\leqq\lambda\leqq\nu(k_0)} \sum_{l=1}^{\lambda} a_{n(l)}\psi_{n(l)}(x) \geqq \tag{9}$$

$$\geqq C_5(K)\left(|a_1|+|a_2|+\sum_{k=0}^{k_0-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}}\right) \qquad (x\in E)$$

erfüllt ist.

Beweis des Hilfssatzes I′. Wir setzen $\psi_n(x) = \operatorname{sign} a_n\cdot r_n(x)$

$$\left(r_n(x)=\operatorname{sign}\sin 2^n\pi x;\quad n=1,\ldots,\nu(C_0(K))\right).$$

Es ist

$$\sum_{n=1}^{\nu(C_0(K))} a_n\psi_n(x) = \sum_{n=1}^{\nu(C_0(K))} |a_n| \qquad \left(x\in(0,1/2^{\nu(C_0(K))})\right). \tag{10}$$

Wir wenden den Hilfssatz I für die Folge $\{a_n\}$ $\left(n=\nu(C_0(K))+1,\ldots,\nu(k_0)\right)$ im Falle $k_1=C_0(K)$, $k_2=k_0$ an. Dann ergibt sich ein K-beschränktes ONS von Treppenfunktionen $\psi_n^*(x)$ $\left(n=\nu(C_0(K))+1,\ldots,\nu(k_0)\right)$, eine einfache Menge $E^*(\subseteqq(0,1))$ mit $m(E^*)\geqq C_1(K)$ derart, daß die Summe $\sum_{n=\nu(C_0(K))+1}^{\nu(k_0)} a_n\psi_n^*(x)$ eine Anordnung $\sum_{l=1}^{\nu(k_0)-\nu(C_0(K))} a_{m(l)}\psi^*_{m(l)}(x)$ besitzt, für die

$$\max_{1\leqq\lambda\leqq\nu(k_0)-\nu(C_0(K))} \sum_{l=1}^{\lambda} a_{m(l)}\psi^*_{m(l)}(x) \geqq \tag{11}$$

$$\geqq C_2(K)\sum_{k=C_0(K)}^{k_0-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}} \qquad (x\in E^*)$$

erfüllt wird. Wir setzen

$$\psi_n(x) = \sum_{s=1}^{2^{\nu(C_0(K))}} \psi_n^*\left(\left(\frac{s-1}{2^{\nu(C_0(K))}},\ \frac{s-1}{2^{\nu(C_0(K))}}+\frac{1}{2^{\nu(C_0(K))+1}}\right);x\right)-$$

$$-\sum_{s=1}^{2^{\nu(C_0(K))}} \psi_n^*\left(\left(\frac{s-1}{2^{\nu(C_0(K))}}+\frac{1}{2^{\nu(C_0(K))+1}},\ \frac{s}{2^{\nu_0(C(K))}}\right);x\right)$$

$\left(n=\nu(C_0(K))+1,\ldots,\nu(k_0)\right)$. Offensichtlich bilden die Treppenfunktionen $\psi_n(x)$ $\left(n=1,\ldots,\nu(k_0)\right)$ ein K-beschränktes ONS. Es sei $E=E^*\left((0,1/2^{\nu(C_0(K))+1})\right)$; E ist einfach, und gilt $m(E)=m(E^*)/2^{\nu(C_0(K))+1}\geqq C_4(K)$ auf Grund des Hilfssatzes I. Wir definieren eine Anordnung der Folge $1,\ldots,\nu(k_0)$; es sei $n(l)=l$

$(l=1, \ldots, \nu(C_0(K))),\ n(l) = m(l-\nu(C_0(K)))\ (l = \nu(C_0(K))+1, \ldots, \nu(k_0))$. Aus (10) und (11) folgt

$$\max_{1 \leqq \lambda \leqq \nu(k_0)} \sum_{l=1}^{\lambda} a_{n(l)} \psi_{n(l)}(x) \geqq \tag{12}$$

$$\geqq C_6(K)\left(\sum_{n=1}^{\nu(C_0(K))} |a_n| + \sum_{k=C_0(K)}^{k_0-1} \left(a_{\nu(k)+1}^2 + \sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2 \log^2 l\right)^{\frac{1}{2}}\right) \qquad (x \in E).$$

Durch einfacher Rechnung erhalten wir

$$\sum_{n=1}^{\nu(C_0(K))} |a_n| + \sum_{k=C_0(K)}^{k_0-1} \left(a_{\nu(k)+1}^2 + \sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2 \log^2 l\right)^{\frac{1}{2}} \geqq$$

$$\geqq C_7(K)\left(|a_1| + |a_2| + \sum_{k=0}^{k_0-1} \left(\sum_{n=\nu(k)+1}^{\nu(k+1)} a_n^2 \log^2 n\right)^{\frac{1}{2}}\right),$$

und so aus (12) ergibt sich (9).

Aus dem Hilfssatz I' erhalten wir (8) unmittelbar. Aus (6), (7) und (8) folgt

$$C_3(K) S_1 \leqq \|\{a_n\};\, K\|^* \leqq A_2 S_1 \qquad (K>1). \tag{3}$$

Eine ähnliche genaue Abschätzung für $\|\{a_n\}; K\|^*$ ist im Falle $K=1$ noch nicht bekannt (s. die Bemerkung in 2). Nach (7) und (13) haben also $\|\{a_n\}; \infty\|^*$ und $\|\{a_n\}; K\|^*$ $(K>1)$ dieselbe „Grössenordnung". Diese Tatsache spricht der folgende, aus (6), (7) und (13) sich ergebende Satz aus.

Satz III. *Ist $K>1$, dann besteht für jede Koeffizientenfolge $\{a_n\}$*

$$C_8(K) \sup_P \sup_{\{\varphi_n\}} \int_0^1 \left(\sup_{1 \leqq i \leqq j} (a_i\varphi_i(x) + \cdots + a_j\varphi_j(x))^2\right) dx \leqq$$

$$\leqq \sup_P \sup_{|\varphi_n| \leqq K} \int_0^1 \left(\sup_{1 \leqq i \leqq j} (a_i\varphi_i(x) + \cdots + a_j\varphi_j(x))^2\right) dx \leqq$$

$$\leqq \sup_P \sup_{\{\varphi_n\}} \int_0^1 \left(\sup_{1 \leqq i \leqq j} (a_i\varphi_i(x) + \cdots + a_j\varphi_j(x))^2\right) dx.$$

5. Zum Beweis des Hilfssatzes I soll es einige Vorbereitungen vorausschicken. In der Arbeit [8] haben wir für eine Koeffizientenfolge $\{b_n\}$ noch eine weitere „Norm" definiert.

Es sei $K \geqq 1$. Für eine endliche Folge $\{b_n\}$ $(n=p, \ldots, q;\ 1 \leqq p \leqq q)$ setzen wir die Funktion

$$I(b_p, \ldots, b_q; K) = \sup_{|\varphi_n| \leqq K} \int_0^1 \left(\max_{p \leqq i \leqq j \leqq q} (b_i\varphi_i(x) + \cdots + b_j\varphi_j(x))^2\right) dx,$$

und für eine beliebige Folge $\{b_n\}$ sei

$$\|\{b_n\}; K\| = \sup \sum_{k=0}^{\infty} I^{\frac{1}{2}}(b_{\mu(k)+1}, \ldots, b_{\mu(k+1)}; K),$$

wobei das Supremum für jede unendliche Indexfolge $(0=)\mu(0)<\cdots<\mu(k)<\cdots$ gebildet wird.

Wir haben bewiesen, daß im Falle $\|\{b_n\}; K\|<\infty$

$$\|\{b_n\}; K\| = \lim_{N\to\infty} I^{\frac{1}{2}}(b_1, \ldots, b_N; K)$$

gilt. (Siehe [8], S. 135.) Da für eine endliche Folge $\{b_n\}$ $(n=1, \ldots. N)$ $\|\{b_n\}; K\|<\infty$ offensichtlich besteht, ergibt sich also

$$\|\{b_n\}_1^N; K\| = I^{\frac{1}{2}}(b_1, \ldots, b_N; K). \tag{14}$$

Weiterhin wurde die folgende Behauptung gezeigt ([8], Satz II): Ist $K>1$ und $|b_1|\geqq\cdots\geqq|b_n|\geqq\cdots$, dann besteht

$$C_9(K)\left(b_1^2+\sum_{n=2}^{\infty} b_n^2\log^2 n\right)^{\frac{1}{2}} < \|\{b_n\}; K\|. \tag{15}$$

Aus (14) und (15) ergibt sich also:

Hilfssatz II. *Ist $K>1$, und gilt $|b_1|\geqq\cdots\geqq|b_N|$, dann besteht*

$$\sup_{|\varphi_n|\leqq K}\left\{\int_0^1\left(\max_{1\leqq i\leqq N}(b_1\varphi_1(x)+\cdots+b_i\varphi_i(x))^2\right)dx\right\}^{\frac{1}{2}} > \tag{16}$$

$$> C_{10}(K)\left(b_1^2+\sum_{l=2}^{N} b_l^2\log^2 l\right)^{\frac{1}{2}} \qquad (C_{10}(K)<1).$$

Hilfssatz III. *Es seien $K>1$ und $\{b_n\}$ $(n=1, \ldots, N)$ eine Zahlenfolge. Dann gibt es ein K-beschränktes ONS von Treppenfunktionen $\psi_n(x)$ $(n=1, \ldots, N)$ und eine einfache Menge $G(\subseteqq(0, 1))$ mit $m(G)\geqq C_{11}(K)$ derart, daß*

$$\max_{1\leqq i\leqq N}\sum_{n=1}^{i} b_n\psi_n(x) > A_3 I^{\frac{1}{2}}(b_1, \ldots, b_N; K) \qquad (x\in G;\quad A_3<1)$$

erfüllt wird.

Beweis des Hilfssatzes III. Mit einer in der Arbeit [8] angewandten Methode bekommen wir ein K-beschränktes ONS von Treppenfunktionen $\chi_n(x)$ $(n=1, \ldots, N)$, für welches

$$A=\left\{\int_0^1\left(\max_{1\leqq i\leqq N}(b_1\chi_1(x)+\cdots+b_i\chi_i(x))^2\right)dx\right\}^{\frac{1}{2}} > \frac{1}{2} I^{\frac{1}{2}}(b_1, \ldots, b_N; K) \tag{17}$$

gilt. Ohne Beschränkung der Allgemeinheit können wir

$$A=\sqrt{2} \tag{18}$$

annehmen; im entgegengesetzten Falle beweisen wir nähmlich den Hilfssatz III für die Folge $\{\sqrt{2}b_n/A\}$, woraus die Behauptung auch für die Folge $\{b_n\}$ sich ergibt. Weiterhin, auch ohne Beschränkung der Allgemeinheit können wir

$$F(x)=\max_{1\leqq i\leqq N}|b_1\chi_1(x)+\cdots+b_i\chi_i(x)|=\max_{1\leqq i\leqq N}(b_1\chi_1(x)+\cdots+b_i\chi_i(x)) \tag{19}$$

annehmen. Im entgegensetzten Falle betrachten wir das System $\chi_n^{(0)}(x)=\chi_n(x)G(x)$ $(n=1, \ldots, N)$, wobei $G(x)$ folgenderweise definiert ist: es sei $i(x)$ die kleinste positive ganze Zahl, für die

$$\max_{1\leqq i\leqq N}|b_1\chi_1(x)+\cdots+b_i\chi_i(x)|=|b_1\chi_1(x)+\cdots+b_{i(x)}\chi_{i(x)}(x)|$$

gilt, und wir setzen

$$G(x)=\operatorname{sign}(b_1\chi_1(x)+\cdots+b_{i(x)}\chi_{i(x)}(x)).$$

Da die Funktionen $\chi_n(x)$ Treppenfunktionen sind, sind auch die Funktionen $\chi_n^{(0)}(x)$ dieselben. Diese Funktionen bilden offensichtlich ein K-beschränktes ONS, und für diese bestehen schon (17) und (19).

Da die Funktionen $\chi_n(x)$ Treppenfunktionen sind, gibt es eine Zerlegung von $(0, 1)$ auf endlich viele Intervalle $J_1, \ldots, J_\varrho$ derart, daß jede Funktion $\chi_n(x)$ in jedem J_r konstant ist. Den Wert von $F(x)$ im Intervall $J_r=(a_r, \bar{a}_r)$ bezeichnen wir mit w_r $(r=1, \ldots, \varrho)$. Aus (17) und (18) folgt

$$\sum_{r=1}^{\varrho} w_r^2 m(J_r)=2. \tag{20}$$

Es seien $(1\leqq)r_1<\cdots<r_k(\leqq\varrho)$ diejenige Indizes $(1\leqq r\leqq\varrho)$, für die $w_r\geqq 1$ gilt; die übrigen Indizes r $(1\leqq r\leqq\varrho)$ bezeichnen wir mit $s_1<\cdots<s_\varkappa$. Aus (20) erhalten wir

$$\sum_{l=1}^{\varkappa} m(J_{s_l})<1, \quad (2\geqq)\sum_{l=1}^{k} w_{r_l}^2 m(J_{r_l})>1. \tag{21}$$

Wir setzen

$$u_l=\sum_{p=1}^{l} w_{r_p}^2 m(J_{r_p}) \qquad (l=0, \ldots, k),$$

$$u_{k+l}=u_k+\sum_{p=1}^{l} m(J_{s_p}) \qquad (l=1, \ldots, \varkappa),$$

$$\bar{\chi}_n(x)=\begin{cases} w_{r_l}^{-1}\chi_n((x-u_{l-1})/w_{r_l}^2+a_{r_l}) & (x\in(u_{l-1}, u_l);\ l=1, \ldots, k),\\ \chi_n(x-u_l+a_{s_l}) & (x\in(u_l, u_{l+1});\ l=k, \ldots, k+\varkappa-1)\end{cases}$$

$(n=1, \ldots, N)$. Offensichtlich bilden die Treppenfunktionen $\bar{\chi}_n(x)$ $(n=1, \ldots, N)$ ein K-beschränktes ONS im Intervall $(0, u_{k+\varkappa})$, weiterhin gilt nach (17) und (18)

$$\max_{1\leqq i\leqq N}\left(b_1\bar{\chi}_1(x)+\cdots+b_i\bar{\chi}_i(x)\right) = 1(=A/\sqrt{2}) \qquad (x\in(0, u_k)). \tag{22}$$

Es sei nun $D(K)$ diejenige positive Zahl, für die

$$D^{-1}(K)u_{k+\varkappa}^{-1}+\left(1-D^{-1}(K)\right)K^2 = 1 \tag{23}$$

besteht. Offensichtlich hat dieser von K, aber auch von der Folge $\{b_n\}$ und von dem System $\{\chi_n(x)\}$ abhängige Wert eine nur von K abhängige obere Schranke:

$$(1<)D(K)\leqq C_{12}(K). \tag{24}$$

Wir setzen

$$\psi_n(x) = \begin{cases} \bar{\chi}_n(D(K)u_{k+\varkappa}x) & (x\in(0, D^{-1}(K))), \\ Kr_n((x-D^{-1}(K))/(1-D^{-1}(K))) & (x\in(D^{-1}(K), 1)) \end{cases}$$

$(n=1, \ldots, N)$. Nach (23) bilden diese Treppenfunktionen ein K-beschränktes ONS. Da nach (21) $u_{k+\varkappa}<3$ ist, folgt es aus (17), (22) und (24), daß alle Erforderungen des Hilfssatzes III mit $G=(0, 1/3D(K))$ und $A_3 = \dfrac{1}{2\sqrt{2}}$ erfüllt sind.

Hilfssatz IV. *Unter der Bedingung des Hilfssatzes III gibt es ein K-beschränktes ONS von Treppenfunktionen $\psi_n(x)$ $(n=1, \ldots, N)$, Indizes $(1\leqq)i(1)\leqq\cdots\leqq i(N)(\leqq N)$ und paarweise disjunkte, einfache Mengen $E_l(\subseteqq(0, 1))$ mit $m(E_l) = C_{11}(K)/16N$ $(l=1, \ldots, N)$ und*

$$\sum_{n=1}^{i(l)} b_n\psi_n(x) > A_3 I^{\frac{1}{2}}(b_1, \ldots, b_N; K) \qquad (x\in E_l;\quad l=1, \ldots, N),$$

wobei A_3 und $C_{11}(K)$ die Konstanten im Hilfssatz III sind.

Beweis des Hilfssatzes IV. Wir brauchen die Bezeichnungen vom Hilfssatz III. Es sei $G'(\subseteqq G)$ eine einfache Menge mit $m(G')=C_{11}(K)$. Weiterhin bezeichnen wir mit E_i' die Menge derjenigen Punkte $x(\in G')$, für die $i(x)=i$ besteht $(i=1, \ldots, N)$. Es seien $(1\leqq)j(1)<\cdots<j(\nu)(\leqq N)$ diejenige Indizes i, für die $m(E_i')\geqq$ $\geqq C_{11}(K)/2N$ gilt. Dann ist

$$\sum_{l=1}^{\nu} m(E_{j(l)}')\geqq C_{11}(K)/2,$$

und gibt es positive ganze Zahlen $\bar{\alpha}_l$ mit

$$\bar{\alpha}_l C_{11}(K)/2N \leqq m(E_{j(l)}') < (\bar{\alpha}_l+1)C_{11}(K)/2N \qquad (l=1, \ldots, \nu).$$

Es sei $E''_{j(l)}$ eine einfache Untermenge von $E'_{j(l)}$ mit $m(E''_{j(l)}) = \bar{\alpha}_l C_{11}(K)/2N$ $(l=1, \ldots, \nu)$. Dann gilt

$$\sum_{l=1}^{\nu} m(E''_{j(l)}) = \frac{C_{11}(K)}{4N} \sum_{l=1}^{\nu} 2\bar{\alpha}_l \geqq C_{11}(K)/4.$$

Nach Verminderung gewisser $2\bar{\alpha}_l$ erhalten wir positive ganze Zahlen α_l $(l=1, \ldots, N)$ mit $\alpha_1 + \cdots + \alpha_\nu = N$. Es sei $\bar{E}_{j(l)}$ eine einfache Untermenge von $E''_{j(l)}$ mit $m(\bar{E}_{j(l)}) = = \alpha_l C_{11}(K)/16N$ $(l=1, \ldots, N)$. Wir teilen die Menge $\bar{E}_{j(l)}$ in α_l einfache Untermenge vom Mass $C_{11}(K)/16N$ $(l=1, \ldots, \nu)$ ein; die soeben sich ergebenden Mengen bezeichnen wir mit E_l $(l=1, \ldots, N)$. Offensichtlich werden die Behauptungen des Hilfssatzes IV mit gewissen $i(l)$ erfüllt.

Hilfssatz V. *Es sei* $K>1$, $\{b_n\}$ $(n=1, \ldots, N)$ *eine Zahlenfolge mit* $|b_1| \geqq \cdots \cdots \geqq |b_N|$ *und*

$$\sqrt{C_{11}(K)/2}\, C_{10}(K) A_3 \left(b_1^2 + \sum_{n=2}^{N} b_n^2 \log^2 n\right)^{\frac{1}{2}} \Big/ 2 \geqq \left(\sum_{n=1}^{N} b_n^2\right)^{\frac{1}{2}}, \tag{25}$$

wobei $C_{10}(K)$ *die Konstante im Hilfssatz II,* A_3 *und* $C_{11}(K)$ *aber die Konstanten im Hilfssatz III bedeuten. Dann gibt es ein K-beschränktes ONS von Treppenfunktionen* $\psi_n(x)$ $(n=1, \ldots, N)$. *Indizes* $(1 \leqq) i(1), \ldots, i(2N) (\leqq N)$ *und eine Zerlegung des Intervalls* $(0, C_{11}(K)/8)$ *auf paarweise disjunkte Intervalle* $I_1, \ldots, I_{2N}$, *für die*

$$\int_0^{C_{11}(K)/8} \psi_n(x)\,dx = 0, \qquad \int_{C_{11}(K)/8}^{1} \psi_n(x)\,dx = 0 \qquad (n=1, \ldots, N), \tag{26}$$

$$m(I_m) = C_{11}(K)/16N \qquad (m=1, \ldots, 2N), \tag{27}$$

und

$$b_1\psi_1(x) + \cdots + b_{i(m)}\psi_{i(m)}(x) > C_{13}(K)\left(b_1^2 + \sum_{n=2}^{N} b_n^2 \log^2 n\right)^{\frac{1}{2}}$$
$$(x \in I_m;\quad m=1, \ldots, N),$$

$$b_{i(m)}\psi_{i(m)}(x) + \cdots + b_N\psi_N(x) > C_{13}(K)\left(b_1^2 + \sum_{n=2}^{N} b_n^2 \log^2 n\right)^{\frac{1}{2}} \tag{28}$$
$$(x \in I_m;\quad m = N+1, \ldots, 2N)$$

erfüllt sind.

Beweis des Hilfssatzes V. Durch Anwendung der Hilfssätze II und III ergibt sich ein K-beschränktes ONS von Treppenfunktionen $\chi_n(x)$ $(n=1, \ldots, N)$ und eine einfache Menge G mit

$$m(G) \geqq C_{11}(K) \tag{29}$$

derart, dass

$$\max_{1 \leqq i \leqq N} \sum_{n=1}^{i} b_n \chi_n(x) > C_{10}(K) A_3 \left(b_1^2 + \sum_{n=2}^{N} b_n^2 \log^2 n \right)^{\frac{1}{2}} \qquad (x \in G) \tag{30}$$

erfüllt ist. Wir setzen zur Abkürzung:

$$B = \left(b_1^2 + \sum_{n=2}^{N} b_n^2 \log^2 n \right)^{\frac{1}{2}}.$$

Es sei

$$G_1 = \left\{ x : 0 \leqq x \leqq 1, \left| \sum_{n=1}^{N} b_n \chi_n(x) \right| > C_{10}(K) A_3 B/2 \right\}.$$

Da

$$m(G_1) C_{10}^2(K) A_3^2 B^2/4 \leqq \int_0^1 \left(\sum_{n=1}^{N} b_n \chi_n(x) \right)^2 dx \leqq \sum_{u=1}^{N} b_n^2$$

gilt, erhalten wir auf Grund von (25)

$$m(G_1) \leqq C_{11}(K)/2. \tag{31}$$

Offensichtlich ist $G_2 = G \setminus G_1$ einfach, und gilt nach (29) und (31)

$$m(G_2) \geqq C_{11}(K)/2. \tag{32}$$

Weiterhin aus (25), (30) und aus der Definition der Menge G_1 erhalten wir:

$$\max_{1 \leqq i < N} \sum_{n=1}^{i} b_n \chi_n(x) > C_{10}(K) A_3 B \qquad (x \in G_2), \tag{33}$$

$$\min_{1 < i \leqq N} \sum_{n=i}^{N} b_n \chi_n(x) < -C_{10}(K) A_3 B/2 \qquad (x \in G_2). \tag{34}$$

Es sei $i(x)$ die kleinste positive ganze Zahl ($< N$), für die

$$\max_{1 \leqq i < N} \sum_{n=1}^{i} b_n \chi_n(x) = \sum_{n=1}^{i(x)} b_n \chi_n(x) \qquad (x \in G_2)$$

besteht; witerhin bezeichnen wir mit $H_i (\subseteqq G_2)$ die Menge derjenigen Punkte x, für die $i(x) = i$ ($i = 1, \ldots, N-1$) ist. Seien $(1 \leqq) j(1) < \cdots < j(\nu) (< N)$ diejenige Indizes i, für die $m(H_i) \geqq C_{11}(K)/4N$ besteht. Weiterhin anstatt der Menge $H_{j(l)}$ nehmen wir eine einfache Menge $\bar{H}_{j(l)} (\subseteqq H_{j(l)})$ mit

$$m(\bar{H}_{j(l)}) = \bar{\alpha}_l C_{11}(K)/4N \qquad (l = 1, \ldots, \nu),$$

wobei die positiven ganzen Zahlen $\bar{\alpha}_l$ mit

$$\bar{\alpha}_l C_{11}(K)/4N \leqq m(H_{j(l)}) < (\bar{\alpha}_l + 1) C_{11}(K)/4N \qquad (l = 1, \ldots, \nu)$$

bestimmt sind. Für die Mengen $\overline{H}_{j(l)}$ gilt

$$\sum_{l=1}^{\nu} m(\overline{H}_{j(l)}) = \frac{C_{11}(K)}{4N} \sum_{l=1}^{\nu} \bar{\alpha}_l = \frac{C_{11}(K)}{8N} \sum_{l=1}^{\nu} 2\bar{\alpha}_l \geqq C_{11}(K)/8$$

auf Grund von (32) und der Definition von $\bar{\alpha}_l$. Nach Verminderung gewisser $2\bar{\alpha}_l$ erhalten wir positive ganze Zahlen α_l $(l=1, \ldots, \nu)$ mit

$$(\alpha_1 + \cdots + \alpha_\nu)/N = 1. \tag{35}$$

Es sei nun F_l eine einfache Untermenge von $\overline{H}_{j(l)}$ mit

$$m(F_l) = \alpha_l C_{11}(K)/8N \qquad (l=1, \ldots, \nu). \tag{36}$$

Die Menge F_l teilen wir in α_l einfache Mengen vom Mass $C_{11}(K)/8N$ $(l=1, \ldots, \nu)$. Nach (35) und (36) bekommen wir also paarweise disjunkte, einfache Mengen; bezeichnen wir diese in Reihe nach mit F_l' $(l=1, \ldots, N)$.

Wir setzen

$$\bar{\psi}_n(x) = \chi_n((0, 1/2); x) - \chi_n((1/2, 1); x) \qquad (n=1, \ldots, N),$$

$$\overline{F}_l = F_l'((0, 1/2)) \qquad (l=1, \ldots, N), \qquad \overline{F}_l = F_l'((1/2, 1)) \qquad (l=N+1, \ldots, 2N).$$

Diese Treppenfunktionen bilden ein K-beschränktes ONS; diese einfachen Mengen sind paarweise disjunkt; nach (33) und (34) gelten

$$\sum_{n=1}^{i(m)} b_n \bar{\psi}_n(x) > C_{10}(K) A_3 B \qquad (x \in \overline{F}_m;\quad m=1, \ldots, N), \tag{37}$$

$$\sum_{n=i(m)}^{N} b_n \bar{\psi}_n(x) > C_{10}(K) A_3 B/2 \qquad (x \in \overline{F}_m;\quad m=N+1, \ldots, 2N) \tag{38}$$

mit gewissen Indizes $i(m)$, weiterhin gelten

$$m(\overline{F}_m) = C_{11}(K)/16N, \tag{39}$$

$$\sum_{m=1}^{2N} m(\overline{F}_m) = C_{11}(K)/8. \tag{40}$$

Die Menge

$$G_3 = (0, 1) \setminus \bigcup_{l=1}^{2N} \overline{F}_l$$

ist auch einfach, also gilt eine Darstellung

$$G_3 = \bigcup_{s=1}^{\sigma} J_s,$$

wobei $\bar{J}_s$ paarweise disjunkte Intervalle sind. Weiterhin gilt

$$\bar{F}_l = \bigcup_{s=1}^{\sigma_l} \bar{I}_s(l) \qquad (l=1, \ldots, 2N)$$

mit paarweise disjunkten Intervallen $\bar{I}_s(l)$. Wir setzen

$$I_m = \left(\sum_{l=1}^{m-1} \sum_{s=1}^{\sigma_l} m(\bar{I}_s(l)),\ \sum_{l=1}^{m} \sum_{s=1}^{\sigma_l} m(\bar{I}_s(l))\right) \qquad (m=1, \ldots, 2N),$$

$$I_s(l) = \left(\sum_{p=1}^{l-1} \sum_{q=1}^{\sigma_p} m(\bar{I}_q(p)) + \sum_{j=1}^{s-1} m(\bar{I}_j(l)),\ \sum_{p=1}^{l-1} \sum_{q=1}^{\sigma_p} m(\bar{I}_q(p)) + \sum_{j=1}^{s} m(\bar{I}_j(l))\right)$$

$(s=1, \ldots, \sigma\ ;\ l=1, \ldots, 2N)$. Weiterhin seien J_s paarweise disjunkte Intervalle mit

$$\bigcup_{s=1}^{\sigma} J_s = \left(C_{11}(K)/8, 1\right), \quad m(J_s) = m(\bar{J}_s) \qquad (s=1, \ldots, \sigma);$$

auf Grund von (40) können wir die Intervalle J_s derart anwählen. Auf Grund von (39) ergibt sich (27), und nach (40) folgt, daß die Intervalle I_m eine Zerlegung des Intervalls $\left(0, C_{11}(K)/8\right)$ bilden.

Mit T bezeichnen wir diejenige umkehrbar eindeutige Transformation, die die Intervalle $\bar{J}_s$ auf J_s und die Intervalle $\bar{I}_s(l)$ auf $I_s(l)$ linear abbildet; die inverze Transformation von T bezeichnen wir mit T^{-1}. Wir setzen endlich

$$\psi_n(x) = \bar{\psi}_n(T^{-1}x) \qquad (n=1, \ldots, N).$$

Offensichtlich bilden diese Treppenfunktionen ein K-beschränktes ONS, und auf Grund von (37), (38) sind auch die Ungleichungen (28) erfüllt, wobei $C_{13}(K) = C_{10}(K)A_3/2$. (26) ist nach der Konstruktion offensichtlich. Damit haben wir Hilfssatz V bewiesen.

Wir werden noch einen Hilfssatz anwenden.

Hilfssatz von Menchoff. ([2], S. 104.) *Es seien d und q positive ganze Zahlen $0<d<q$. Zu jedem Indexpaar (i,j) mit $1 \leqq i, j \leqq q$ und $|i-j| = d$ soll eine von Null verschiedene Zahl $\alpha_{i,j}$ zugeordnet werden; wir bezeichnen mit β_d das Maximum der absoluten Beträge der Zahlen $\alpha_{i,j}$. In jedem Intervall (u, v) mit $v-u > 2\beta_d$ können dann Treppenfunktionen $\varphi_l(x)$ $(l=1, \ldots, q)$ derart definiert werden, daß die folgenden Bedingungen erfüllt sind:*

$$|\varphi_l(x)| = 1 \qquad (u<x<v;\quad l=1, \ldots, q),$$

$$\int_u^v \varphi_i(x)\varphi_j(x)\,dx = -\alpha_{i,j} \qquad (|i-j| = d;\quad 1 \leqq i, j \leqq q),$$

$$\int_u^v \varphi_i(x)\varphi_j(x)\,dx = 0 \qquad (i \neq j;\quad |i-j| \neq d;\quad 1 \leqq i, j \leqq q).$$

Die positive ganze Zahl $C_0(K)$, die im Hilfssatz I erwähnt wurde, bestimmen wir folgenderweise: $C_0(K)$ sei die kleinste positive ganze Zahl, für die

$$\sqrt{C_{11}(K)}\, C_{10}(K) A_3 2^{C_0(K)}/4 \geqq 1 \tag{41}$$

besteht (für die Bedeutung der Konstanten A_3, $C_{10}(K)$, $C_{11}(K)$ siehe den Hilfssatz V).

Bei dem Beweis des Hilfssatzes I werden wir zwei Fälle unterscheiden.

6. Erstens nehmen wir

(42)
$$\frac{1}{2}\sum_{k=k_1+1}^{k_2-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}} \leqq \sum_{k=k_1+1}^{k_2-1}|a_{\nu(k+1)}|\left(1+\sum_{l=2}^{\nu(k+1)-\nu(k)}\log^2 l\right)^{\frac{1}{2}}$$

an. Wir setzen

$$c_n = a_n \quad (n = \nu(k_1)+1, \dots, \nu(k_1+1)),$$
$$c_n = a_{\nu(k+1)} \quad (n = \nu(k)+1, \dots, \nu(k+1);\ \ k = k_1+1, \dots, k_2-1).$$

Aus (42) folgt

$$\frac{1}{2}\sum_{k=k_1}^{k_2-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}} \leqq \tag{43}$$
$$\leqq \sum_{k=k_1}^{k_2-1}\left(c^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} c^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}}.$$

Dann gibt es eine ganze Zahl s $(0 \leqq s \leqq 2)$ mit

$$\sum_{\substack{i\\ k_1 \leqq 3i+s \leqq k_2-1}}\left(c^2_{\nu(3i+s)+1}+\sum_{l=2}^{\nu(3i+s+1)-\nu(3i+s)} c^2_{\nu(3i+s)+l}\log^2 l\right)^{\frac{1}{2}} \geqq \tag{44}$$
$$\geqq \frac{1}{3}\sum_{k=k_1}^{k_2-1}\left(c^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} c^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}};$$

die Indizes $(k_1 \leqq)\, 3i+s\, (<k_2)$ bezeichnen wir der Reihe nach mit $r_1 < \dots < r_{j_0}$, weiterhin sei $r_0 = k_1$, $r_{j_0+1} = k_2$.

Durch vollständige Induktion werden wir zeigen, daß für jedes i $(1 \leqq i \leqq j_0)$ kann man ein K-beschränktes ONS von Treppenfunktionen $\psi_n(x)$ $(n = \nu(r_0)+1, \dots, \nu(r_{i+1}))$ im Intervall $(0, 3)$ mit folgenden Eigenschaften bilden: gibt es paarweise disjunkte, einfache Mengen $E_l(i) (\subseteqq (0, 1))$ $(l = 1, \dots, \nu(r_i+1)-\nu(r_i))$ mit

$$m(E_l(i)) = C_{11}(K)/16(\nu(r_i+1)-\nu(r_i)) \qquad (l = 1, \dots, \nu(r_i+1)-\nu(r_i)), \tag{45}$$

weiterhin gibt es eine Anordnung

$$\sum_{t=1}^{\nu(r_{i+1})-\nu(r_0)} c_{n(t,i)}\psi_{n(t,i)}(x)$$

der Summe $\sum_{n=r_0+1}^{\nu(r_{i+1})} c_n\psi_n(x)$ derart, daß

$$(46)\qquad \sum_{t=i'(l,i)}^{i''(l,i)} c_{n(t,i)}\psi_{n(t,i)}(x) > \frac{C_{13}(K)}{32}\sum_{j=1}^{i}\left[c^2_{\nu(r_j)+1}+\sum_{l=2}^{\nu(r_{j+1})-\nu(r_j)} c^2_{\nu(r_j)+l}\log^2 l\right]^{\frac{1}{2}}$$

$$(x\in E_l(i))$$

mit gewissen Indizes

$$(1\leqq)\, i'(l,i) \leqq i''(l,i)(\leqq \nu(r_{i+1})-\nu(r_0)) \qquad (l=1,\dots,\nu(r_i+1)-\nu(r_i))$$

besteht. $(C_{13}(K) = C_{10}(K)A_3/2)$.

Es sei

$$\psi_n(x) = \begin{cases} r_n(x-2) & (x\in(2,3)), \\ 0 & \text{sonst} \end{cases} \quad (n=\nu(r_0)+1,\dots,\nu(r_1)).$$

(Im Falle $r_0=r_1$ ist dieser Schritt überflüssig.) Dann wenden wir den Hilfssatz IV mit $b_n=c_{\nu(r_1)+n}$ $(n=1,\dots,\nu(r_1+1)-\nu(r_1))$. So bekommen wir ein K-beschränktes ONS von Treppenfunktionen $\bar\Phi_n(x)$ $(n=\nu(r_1)+1,\dots,\nu(r_1+1))$ im Intervall $(0,1)$ und paarweise disjunkte einfache Mengen $E_l(1)(\subseteqq(0,1))$ mit $m(E_l(1)) = C_{11}(K)/16(\nu(r_1+1)-\nu(r_1))$ $(l=1,\dots,\nu(r_1+1)-\nu(r_1))$ derart, daß mit gewissen Indizes $i(l)$ $(\nu(r_1)<i(l)\leqq\nu(r_1+1))$

$$\sum_{n=\nu(r_1)+1}^{i(l)} c_n\bar\Phi_n(x) > A_3 I^{\frac{1}{2}}(c_{\nu(r_1)+1},\dots,c_{\nu(r_1+1)};K)$$

$$(x\in E_l(1);\quad l=1,\dots,\nu(r_1+1)-\nu(r_1)),$$

oder nach dem Hilfssatz II

$$(47)\qquad \sum_{n=\nu(r_1)+1}^{i(l)} c_n\bar\Phi_n(x) > C_{10}(K)A_3\left[c^2_{\nu(r_1)+1}+\sum_{l=2}^{\nu(r_1+1)-\nu(r_1)} c^2_{\nu(r_1)+l}\log^2 l\right]^{\frac{1}{2}}$$

$$(x\in E_l(1);\quad l=1,\dots,\nu(r_1+1)-\nu(r_1))$$

erfüllt sind. Wir setzen

$$\psi_n(x) = \begin{cases} \bar\Phi_n(x)/\sqrt{3} & (x\in(0,1)), \\ r_n(x-1)/\sqrt{3} & (x\in(1,2)), \\ r_n(x-2)/\sqrt{3} & (x\in(2,3)) \end{cases} \quad (n=\nu(r_1)+1,\dots,\nu(r_1+1)),$$

$$\psi_n(x) = \begin{cases} r_n(x-2) & (x\in(2,3)), \\ 0 & \text{sonst} \end{cases} \quad (n=\nu(r_1+1)+1,\dots,\nu(r_2)).$$

Die Treppenfunktionen $\psi_n(x)$ $\big(n = \nu(r_0)+1, \ldots, \nu(r_2)\big)$ bilden offensichtlich ein K-beschränktes ONS in $(0, 3)$; und aus (47), wegen $C_{13}(K) = \dfrac{C_{10}(K)A_3}{2}$ folgt

$$\sum_{n=\nu(r_1)+1}^{i(l)} c_n\psi_n(x) > C_{13}(K)\left(c^2_{\nu(r_1)+1} + \sum_{l=2}^{\nu(r_1+1)-\nu(r_1)} c^2_{\nu(r_1)+l}\log^2 l\right)^{\frac{1}{2}} >$$

$$> \frac{C_{13}(K)}{32}\left(c^2_{\nu(r_1)+1} + \sum_{l=2}^{\nu(r_2+1)-\nu(r_1)} c^2_{\nu(r_1)+l}\log^2 l\right)^{\frac{1}{2}}$$

$$\big(x\in E_l(1);\quad l = 1, \ldots, \nu(r_1+1)-\nu(r_1)\big)$$

mit gewissen Indizes $i(l)$ $\big(\nu(r_1) < i(l) \leqq \nu(r_1+1)\big)$. Damit haben wir die Treppenfunktionen $\psi_n(x)$ $\big(n = \nu(r_0)+1, \ldots, \nu(r_2)\big)$ und die paarweise disjunkte einfache Mengen $E_l(1)$ $\big(\subseteqq(0, 1)\big)$ derart definiert, daß diese Funktionen ein K-beschränktes ONS in $(0, 3)$ bilden, weiterhin (45) und (46) — in origineller Anordnung — im Falle $i=1$ erfüllt werden.

Es sei i $(1\leqq i<j_0)$ ein Index, und wir nehmen an, daß die Treppenfunktionen $\psi_n(x)$ $\big(n = \nu(r_0)+1, \ldots, \nu(r_{i+1})\big)$ und die paarweise disjunkte einfache Mengen $E_l(i)\big(\subseteqq(0, 1)\big)$ $\big(l = 1, \ldots, \nu(r_i+1)-\nu(r_i)\big)$ schon definiert sind, derart, daß diese Funktionen ein K-beschränktes ONS in $(0, 3)$ bilden, weiterhin (45) und (46) in gewisser Anordnung für i erfüllt werden.

Es seien

$$\begin{aligned}
&\bar N = \big(\nu(r_{i+1}+1)-\nu(r_{i+1})\big)/2\big(\nu(r_i+1)-\nu(r_i)\big)\nu(r_i+1),\\
(48)\qquad &M = \big(\nu(r_{i+1})-\nu(r_{i+1}-1)\big)/2\big(\nu(r_i+1)-\nu(r_i)\big)\nu(r_i+1),\\
&P = \big(\nu(r_{i+1}+1)-\nu(r_{i+1})\big)/\big(\nu(r_{i+1})-\nu(r_{i+1}-1)\big);
\end{aligned}$$

diese sind ganze Zahlen, und gilt $PM=\bar N$. Es sei

$$b_n = c_{\nu(r_{i+1})+(p-1)(\nu(r_{i+1})-\nu(r_{i+1}-1))+1} \qquad \big(n=(p-1)M+1, \ldots, pM;\quad p=1, \ldots, P\big).$$

(In diesem Falle sind die Koeffizienten c_n $\big(n = \nu(r_{i+1})+1, \ldots, \nu(r_{i+1}+1)\big)$ gleich. Solche Definition von b_n ist darum zweckmässig, weil wir den Fall (79) ganz ähnlicherweise betrachten können; dann werden aber nur die Koeffizienten c_n $\big(n = \nu(r_{i+1})+(p-1)\big(\nu(r_{i+1})-\nu(r_{i+1}-1)\big)+1, \ldots, \nu(r_{i+1})+p\big(\nu(r_{i+1})-\nu(r_{i+1}-1)\big)\big)$ gleich $(p=1, \ldots, P)$; im Folgenden werden wir nur diese Tatsache ausnützen.) Wir wenden den Hilfssatz V für die Folge $\{b_n\}$ $(n=1, \ldots, \bar N)$ im Falle $N=\bar N$ an.

Auf Grund der Definition der Folgen $\{c_n\}$, $\{b_n\}$ und M gilt

$$\sqrt{C_{11}(K)/2}\,C_{10}(K)A_3\left(b_1^2+\sum_{n=2}^{N} b_n^2\log^2 n\right)^{\frac{1}{2}}\Big/2 \geqq \sqrt{C_{11}(K)/2}\,C_{10}(K)A_3\cdot$$

$$\cdot\left(b_1^2+\sum_{n=2}^{M/2} b_n^2\log^2 n+\log^2\frac{M}{2}\sum_{n=M/2+1}^{M} b_n^2+\log^2 M\sum_{n=M+1}^{N} b_n^2\right)^{\frac{1}{2}}\Big/2 \geqq$$

$$\geqq \sqrt{C_{11}(K)/2}\,C_{10}(K)A_3\left(\frac{1}{2}\log^2\frac{M}{2}\sum_{n=1}^{M} b_n^2+\log^2 M\sum_{n=M+1}^{N} b_n^2\right)^{\frac{1}{2}}\Big/2 \geqq$$

$$\geqq \sqrt{C_{11}(K)}\,C_{10}(K)A_3\frac{1}{4}\log\frac{M}{2}\left(\sum_{n=1}^{N} b_n^2\right)^{\frac{1}{2}} >$$

$$> \sqrt{C_{10}(K)}\,C_{10}(K)A_3\frac{1}{4}2^{C_0(K)}\left(\sum_{n=1}^{N} b_n^2\right)^{\frac{1}{2}},$$

und wegen (41) besteht (25). Den Hilfssatz V können wir also anwenden. Dann bekommen wir ein K-beschränktes ONS von Treppenfunktionen $\chi_n(x)$ $(n=1, \ldots, \overline{N})$ und paarweise disjunkte Intervalle $I_m(\subseteqq(0, C_{11}(K)/8)$ $(m=1, \ldots, 2\overline{N})$ derart, daß

$$\text{(49)}\qquad \int_0^{C_{11}(K)/8} \chi_n(x)\,dx = 0, \qquad \int_{C_{11}(K)/8}^{1} \chi_n(x)\,dx = 0 \qquad (n=1, \ldots, \overline{N}),$$

$$\text{(50)}\qquad m(I_m) = C_{11}(K)/16\overline{N} \qquad (m=1, \ldots, 2\overline{N})$$

und

$$b_1\chi_1(x)+\cdots+b_{i(m)}\chi_{i(m)}(x) > C_{13}(K)\left(b_1^2+\sum_{n=2}^{N} b_n^2\log^2 n\right)^{\frac{1}{2}}$$
$$(x\in I_m;\quad m=1, \ldots, \overline{N}),$$

$$\text{(51)}\qquad b_{i(m)}\chi_{i(m)}(x)+\cdots+b_N\chi_N(x) > C_{13}(K)\left(b_1^2+\sum_{n=2}^{N} b_n^2\log^2 n\right)^{\frac{1}{2}}$$
$$(x\in I_m;\quad m=\overline{N}+1, \ldots, 2\overline{N})$$

mit gewissen Indizes $i(m)$ $(1\leqq i(m)\leqq \overline{N})$ erfüllt werden.

Wir setzen

$$\psi_n^{(1)}(x)=\begin{cases}\chi_n(x) & (x\in(0, C_{11}(K)/8)),\\ 0 & \text{sonst},\end{cases}\qquad \psi_n^{(2)}(x)=\begin{cases}\chi_n(x) & (x\in(C_{11}(K)/8, 1),\\ 0 & \text{sonst}\end{cases}$$

$(n=1, \ldots, \overline{N})$. Da die Funktionen $\psi_n(x)$ $(n = \nu(r_0)+1, \ldots, \nu(r_{i+1}))$ Treppenfunktionen, und die Mengen $E_l(i)$ $\big(l = 1, \ldots, \nu(r_i+1)-\nu(r_i)\big)$ einfach sind, gibt es Zerlegungen

$$E_l(i) = \bigcup_{p=1}^{q_l} I_p(l) \qquad \big(l = 1, \ldots, \nu(r_i+1)-\nu(r_i)\big),$$

$$(1,2) = \bigcup_{p=1}^{q} J_p, \quad (2,3) = \bigcup_{p=1}^{\bar q} \bar J_p$$

auf paarweise disjunkte Intervalle derart, daß in jedem Teilintervall jede Funktion $\psi_n(x)$ $\big(n = \nu(r_0)+1, \ldots, \nu(r_{i+1})\big)$ konstant ist. Für jedes p seien $J_p(l)$ paarweise disjunkte Teilintervalle von J_p mit der Länge $\dfrac{m(J_p)}{\nu(r_i+1)-\nu(r_i)}$. Es sei weiterhin $J'_p(l)(\subseteqq J_p(l))$ ein Intervall mit

$$m\big(J'_p(l)\big) = C_{11}(K)m(J_p)/16\big(\nu(r_i+1)-\nu(r_i)\big)$$

$\big(p=1, \ldots, q;\ l = 1, \ldots, \nu(r_i+1)-\nu(r_i)\big)$. Wir setzen die Funktionen

$$\bar\psi_{2(l-1)\bar N\nu(r_i+1)+(v-1)\nu(r_i+1)+n}(x) = \sum_{p=1}^{q_l} \psi_v^{(1)}(I_p(l); x) + \sum_{p=1}^{q} \psi_v^{(2)}(J'_p(l); x)$$

$$\big(n = 1, \ldots, \nu(r_i+1)\big),$$

$$\bar\psi_{2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)+(v-1)\nu(r_i+1)+n}(x) = \sum_{p=1}^{q_l} \psi_v^{(1)}(I_p(l); x) + \sum_{p=1}^{q} \psi_v^{(2)}(J'_p(l); x)$$

$$\big(n = 1, \ldots, \nu(r_i+1)\big)$$

$\big(l = 1, \ldots, \nu(r_i+1)-\nu(r_i);\ v = 1, \ldots, \bar N\big)$. Nach (49) ist

$$\int_0^2 \bar\psi_k(x)\psi_l(x)\,dx = 0 \qquad \big(1 \leqq k \leqq \nu(r_{i+1}+1)-\nu(r_{i+1});\ \nu(r_0) < l \leqq \nu(r_{i+1})\big). \tag{52}$$

Offensichtlich gilt

$$\int_0^2 \bar\psi_t(x)\bar\psi_\tau(x)\,dx = 0 \tag{52}$$

$$\big(2(l-1)\bar N\nu(r_i+1) < t \leqq 2l\bar N\nu(r_i+1),\ 2(\lambda-1)\bar N\nu(r_i+1) < \tau \leqq 2\lambda\bar N\nu(r_i+1),\ l \neq \lambda\big).$$

Für ein l bilden wir

$$\alpha_{t,\tau}(l) = \int_0^2 \bar\psi_t(x)\bar\psi_\tau(x)\,dx \qquad \big(2(l-1)\bar N\nu(r_i+1) < t,\ \tau \leqq 2lN\nu(r_i+1)\big).$$

Dann gilt

(54)
$$\alpha_{t,\tau}(l)=\begin{cases} C_{11}(K)/8(\nu(r_i+1)-\nu(r_i)) \\ \quad (t,\tau\in(2(l-1)\bar{N}\nu(r_i+1)+(v-1)\nu(r_i+1),2(l-1)\bar{N}\nu(r_i+1)+v\nu(r_i+1)]\cap \\ \quad \cap(2(l-1)\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1)+(v-1)\nu(r_i+1),\ 2(l-1)\bar{N}\nu(r_i+1)+ \\ \quad +\bar{N}\nu(r_i+1)+v\nu(r_i+1)];\ v=1,\dots,\bar{N}), \\ 0 \quad \text{sonst.}\end{cases}$$

Da $\dfrac{C_{11}(K)3\nu(r_i+1)}{8(\nu(r_i+1)-\nu(r_i))}<1$ ist, können wir auf Grund des Menchoffschen Hilfssatzes Treppenfunktionen $\varphi_n(x)$ $\big(n=1,\dots,\nu(r_{i+1}+1)-\nu(r_{i+1})\big)$ in $(2,3)$ definieren, derart, daß

(55) $$|\varphi_n(x)|=1 \qquad \big(x\in(2,3);\quad n=2(l-1)\bar{N}\nu(r_i+1)+1,\dots,2l\bar{N}\nu(r_i+1)\big),$$

(56) $$\int_2^3\varphi_t(x)\varphi_\tau(x)\,dx=-\alpha_{t,\tau}(l)$$
$$\big(2(l-1)\bar{N}\nu(r_i+1)<t,\ \tau\leqq 2l\bar{N}\nu(r_i+1)\big)$$

erfüllt sind $\big(l=1,\dots,\nu(r_i+1)-\nu(r_i)\big)$.

Durch Rekursion definieren wir Treppenfunktionen
$$\bar{\varphi}_n(x) \qquad \big(n=1,\dots,\nu(r_{i+1}+1)-\nu(r_{i+1})\big),$$
für die

(57) $$|\bar{\varphi}_n(x)|=1 \qquad \big(x\in(2,3);\ n=1,\dots,2l\bar{N}\nu(r_i+1)\big),$$

(58)
$$\int_2^3\bar{\varphi}_t(x)\bar{\varphi}_\tau(x)\,dx=\begin{cases}-\alpha_{t,\tau}(\lambda) & \big(2(\lambda-1)\bar{N}\nu(r_i+1)<t,\ \tau\leqq 2\lambda\bar{N}\nu(r_i+1);\ \lambda=1,\dots,l\big),\\ 0 & \big(2(\lambda-1)\bar{N}\nu(r_i+1)<t\leqq 2\lambda\bar{N}\nu(r_i+1),\\ & \ 2(\bar{\lambda}-1)\bar{N}\nu(r_i+1)<\tau\leqq 2\bar{\lambda}\bar{N}\nu(r_i+1);\\ & \ \lambda\neq\bar{\lambda},\ 1\leqq\lambda,\ \bar{\lambda}\leqq l\big),\end{cases}$$

(59) $$\int_2^3\psi_k(x)\bar{\varphi}_n(x)\,dx=0 \quad \big(\nu(r_0)<k\leqq\nu(r_{i+1});\ 1\leqq n\leqq 2l\bar{N}\nu(r_i+1)\big)$$

für jedes l $\big(l=1,\dots,\nu(r_i+1)-\nu(r_i)\big)$ erfüllt sind.

Die zwei Hälfte von $\bar{J}_p$ bezeichnen wir mit $\bar{J}_p'$, bzw. mit $\bar{J}_p''$ $(p=1,\dots,\bar{q})$. Wir setzen
$$\bar{\varphi}_n(x)=\sum_{p=1}^{\bar{q}}\varphi_n(\bar{J}_p';x)-\sum_{p=1}^{\bar{q}}\varphi_n(\bar{J}_p'';x) \qquad \big(n=1,\dots,2\bar{N}\nu(r_i+1)\big).$$

Aus (55) und (56) folgt, daß (57), (58), (59) für $l=1$ erfüllt sind. Es sei l_0 $\big(1\leqq l_0<\nu(r_i+1)-\nu(r_i)\big)$ eine ganze Zahl. Wir nehmen an, daß die Treppenfunktionen $\bar{\varphi}_n(x)$ $\big(n = 1, \ldots, 2l_0\bar{N}\nu(r_i+1)\big)$ schon definiert sind derart, daß (57), (58), (59) für $l=l_0$ erfüllt sind. Dann gibt es für jedes p $(p=1, \ldots, \bar{q})$ eine Zerlegung

$$\bar{J}_p = \bigcup_{s=1}^{\sigma_p} J_s^*(p)$$

auf paarweise disjunkte Intervalle derart, daß jede Funktion

$$\bar{\varphi}_n \ \big(n = 1, \ldots, 2l_0\bar{N}\nu(r_i+1)\big)$$

in jedem $J_s^*(p)$ konstant ist; die zwei Hälfte von $J_s^*(p)$ bezeichnen wir mit $J_s^{*\prime}(p)$, bzw. mit $J_s^{*\prime\prime}(p)$). Wir setzen

$$\bar{\varphi}_n(x) = \sum_{p=1}^{\bar{q}} \sum_{s=1}^{\sigma_p} \varphi_n\big(J_s^{*\prime}(p); x\big) - \sum_{p=1}^{\bar{q}} \sum_{s=1}^{\sigma_p} \varphi_n\big(J_s^{*\prime\prime}(p); x\big)$$

$$\big(n = 2l_0\bar{N}\nu(r_i+1)+1, \ldots, 2(l_0+1)\bar{N}\nu(r_i+1)\big).$$

Auf Grund von (55) und (56) erhalten wir, daß (57), (58), (59) auch für $l = l_0+1$ bestehen. Durch vollständiger Induktion bekommen wir also Treppenfunktionen $\bar{\varphi}_n(x)$ $\big(n = 1, \ldots, \nu(r_{i+1}+1)-\nu(r_{i+1})\big)$ für die (57), (58) und (59) bei jedem $l = 1, \ldots$ $\ldots, \nu(r_i+1)-\nu(r_i)$ erfüllt werden.

Wir setzen

$$\tilde{\psi}_n(x) = \begin{cases} \bar{\psi}_n(x) & (x \in (0,2)), \\ \bar{\varphi}_n(x) & (x \in (2,3)) \end{cases} \qquad \big(n=1, \ldots, \nu(r_{i+1}+1)-\nu(r_{i+1})\big).$$

Auf Grund von (52), (53), (54), (57), (58) und (59) folgt

$$\text{(60)} \qquad \int_0^3 \psi_k(x)\tilde{\psi}_l(x)\,dx = 0 \qquad \big(\nu(r_0)<k\leqq \nu(r_{i+1});\ 1\leqq l\leqq \nu(r_{i+1}+1)-\nu(r_{i+1})\big),$$

$$\text{(61)} \qquad \int_0^3 \tilde{\psi}_k(x)\tilde{\psi}_l(x)\,dx = \begin{cases} 0 \quad \big(k\neq l;\ 1\leqq k, l\leqq \nu(r_{i+1}+1)-\nu(r_{i+1})\big), \\ 1+C_{11}(K)/8\big(\nu(r_i+1)-\nu(r_i)\big) \\ \qquad \big(k=l;\ 1\leqq k, l\leqq \nu(r_{i+1}+1)-\nu(r_{i+1})\big). \end{cases}$$

Es sei $n(k)$ $\big(k = 1, \ldots, \nu(r_{i+1}+1)-\nu(r_{i+1})\big)$ eine Anordnung der Folge $\nu(r_{i+1})+1, \ldots, \nu(r_{i+1}+1)$, die später definiert wird. Wir setzen endlich

$$\psi_{n(k)}(x) = \tilde{\psi}_k(x)/\sqrt{1+C_{11}(K)/8\big(\nu(r_i+1)-\nu(r_i)\big)} \qquad \big(k=1, \ldots, \nu(r_{i+1}+1)-\nu(r_{i+1})\big).$$

Aus (60) und (61) folgt, daß die Treppenfunktionen

$$\psi_n(x) \ \big(n = \nu(r_0)+1, \ldots, \nu(r_{i+1}+1)\big)$$

ein K-beschränktes ONS im Intervall (0, 3) bilden.

Mit $T(p, l)$ bezeichnen wir diejenige lineare Transformation, die das Intervall $(0, C_{11}(K)/8)$ auf $I_p(l)$ abbildet; die Bildmenge von I_m mit dieser Transformation bezeichnen wir mit $I_p(l, m)$. Es sei

$$\bar{E}_{(l-1)2\bar{N}+m} = \bigcup_{p=1}^{q_l} I_p(l; m) \qquad (l = 1, \ldots, \nu(r_i+1)-\nu(r_i); \; m = 1, \ldots, 2\bar{N}).$$

Nach (45) und (50) ist

$$m(\bar{E}_{(l-1)2\bar{N}+m}) = C_{11}(K)/32\bar{N}\big(\nu(r_i+1)-\nu(r_i)\big).$$

Wir teilen jede einfache Menge $\bar{E}_{(l-1)2\bar{N}+m}$ in $\nu(r_i+1)$ paarweise disjunkte einfache Teilmenge vom gleichen Mass; diese bezeichnen wir der Reihe nach mit

$$E_{(l-1)2\bar{N}\nu(r_i+1)+(m-1)\nu(r_i+1)+s}(i+1)$$

$$(l = 1, \ldots, \nu(r_i+1)-\nu(r_i); \;\; m = 1, \ldots, 2\bar{N}; \;\; s = 1, \ldots, \nu(r_i+1)).$$

Auf Grund von (48) ergibt sich

$$m(E_l(i+1)) = C_{11}(K)/16\big(\nu(r_{i+1}+1)-\nu(r_{i+1})\big) \qquad (l = 1, \ldots, \nu(r_{i+1}+1)-\nu(r_{i+1})).$$

Also ist für die einfachen, paarweise disjunkten Mengen $E_l(i+1)$ (45) auch für $i+1$ erfüllt. Weiterhin aus (51) und aus der Definition der Funktionen $\psi_n(x)$ $\big(\nu(r_{i+1}) < n \leqq \nu(r_{i+1}+1)\big)$ und der Mengen $E_l(i+1)$ $\big(1 \leqq l \leqq \nu(r_{i+1}+1)-\nu(r_{i+1})\big)$ wegen $\sqrt{1+C_{11}(K)/8\big(\nu(r_i+1)-\nu(r_i)\big)} < 2$ folgt, daß

(62)

$$\sum_{v=1}^{j'(l,m)} b_v \psi_{n(2(l-1)\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1)+(v-1)\nu(r_i+1)+1)}(x) \geqq \frac{C_{13}(K)}{2}\left(b_1^2+\sum_{n=2}^{N} b_n^2 \log^2 n\right)^{\frac{1}{2}}$$

$$\left(x \in \bigcup_{m=1}^{\bar{N}\nu(r_i+1)} E_{(l-1)2\bar{N}\nu(r_i+1)+m}(i+1)\right),$$

(63)

$$\sum_{v=j''(l,m)}^{N} b_v \psi_{n(2(l-1)\bar{N}\nu(r_i+1)+(v-1)\nu(r_i+1)+1)}(x) \geqq \frac{C_{13}(K)}{2}\left(b_1^2+\sum_{n=2}^{N} b_n^2 \log^2 n\right)^{\frac{1}{2}}$$

$$\left(x \in \bigcup_{m=\bar{N}\nu(r_i+1)+1}^{2\bar{N}\nu(r_i+1)} E_{(l-1)2\bar{N}\nu(r_i+1)+m}(i+1)\right)$$

mit gewissen Indizes $1 \leqq j'(l, m), j''(l, m) \leqq \bar{N}$ $\big(l = 1, \ldots, \nu(r_i+1)-\nu(r_i)\big)$ erfüllt sind.

Aus der Definition der Folge $\{b_n\}$ und aus (48) ergibt sich durch einfacher Rechnung

(64)
$$16\nu(r_i+1)\left(b_1^2+\sum_{n=2}^{\bar N} b_n^2\log^2 n\right)^{\frac{1}{2}} \geqq$$
$$\geqq \left(c^2_{\nu(r_{i+1})+1}+\sum_{l=2}^{\nu(r_{i+1}+1)-\nu(r_{i+1})} c^2_{\nu(r_{i+1})+l}\log^2 l\right)^{\frac{1}{2}} = C_{i+1}.$$

Wir setzen

(65)
$$n(k) = (p-1)\big(\nu(r_{i+1})-\nu(r_{i+1}-1)+2(l-1)M\nu(r_i+1)+j$$

für

$$k = 2(l-1)\bar N\nu(r_i+1)+2(p-1)M\nu(r_i+1)+j$$
$$\big(l=1, \ldots, \nu(r_i+1)-\nu(r_i);\ p = 1, \ldots, P;\ j = 1, \ldots, 2M\nu(r_i+1)\big).$$

Nach (48), weiterhin nach der Definition der Folgen $\{b_n\}$, $\{n(k)\}$ und von $\psi_n(x)$ erhalten wir aus (62), (63) und (64), daß

$$\sum_{k=2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)+1}^{2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)+j'(l,m)\nu(r_i+1)} c_{n(k)}\psi_{n(k)}(x) =$$
$$= \nu(r_i+1)\sum_{v=1}^{j'(l,m)} b_v\psi_{n(2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)+(v-1)\nu(r_i+1)+1)}(x) \geqq \frac{C_{13}(K)}{32}C_{i+1}$$
$$\left(x\in\bigcup_{m=1}^{\bar N\nu(r_i+1)} E_{(l-1)2\bar N\nu(r_i+1)+m}(i+1)\right),$$

$$\sum_{k=2(l-1)\bar N\nu(r_i+1)+j''(l,m)\nu(r_i+1)+1}^{2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)} c_{n(k)}\psi_{n(k)}(x) =$$
$$= \nu(r_i+1)\sum_{v=j''(l,m)}^{\bar N} b_v\psi_{n(2(l-1)\bar N\nu(r_i+1)+(v-1)\nu(r_i+1)+1)}(x) \geqq \frac{C_{13}(K)}{32}C_{i+1}$$
$$\left(x\in\bigcup_{m=\bar N\nu(r_i+1)+1}^{2\bar N\nu(r_i+1)} E_{(l-1)2\bar N\nu(r_i+1)+m}(i+1)\right),$$

also sind

(66)
$$\sum_{k=2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)+1}^{i_m(l)} c_{n(k)}\psi_{n(k)}(x) > \frac{C_{13}(K)}{32}C_{i+1}$$
$$\left(x\in\bigcup_{m=1}^{\bar N\nu(r_i+1)} E_{(l-1)2\bar N\nu(r_i+1)+m}(i+1)\right),$$

(67)
$$\sum_{k=i_m^*(l)}^{2(l-1)\bar N\nu(r_i+1)+\bar N\nu(r_i+1)} c_{n(k)}\psi_{n(k)}(x) > \frac{C_{13}(K)}{32}C_{i+1}$$
$$\left(x\in\bigcup_{m=\bar N\nu(r_i+1)+1}^{2\bar N\nu(r_i+1)} E_{(l-1)2\bar N\nu(r_i+1)+m}(i+1)\right)$$

mit gewissen Indizes $2(l-1)\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1) < i_m(l) \leqq 2l\bar{N}\nu(r_i+1)$,

$2(l-1)\bar{N}\nu(r_i+1) < i_m^*(l) \leqq 2(l-1)\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1)$ erfült sind.

Da die Funktionen $\psi_n(x)$ $(n = \nu(r_0)+1, \ldots, \nu(r_{i+1}+1))$ Treppenfunktionen sind, gibt es eine Zerlegung

$$(2,3) = \bigcup_{t=1}^{\tau} \tilde{I}_t$$

des Intervalls (2, 3) auf paarweise disjunkte Intervalle $\tilde{I}_t$ derart, daß jede Funktion $\psi_n(x)$ $(n = \nu(r_0)+1, \ldots, \nu(r_{i+1}+1)$ in jedem $\tilde{I}_t$ konstant ist. Wir setzen endlich

$$\psi_n(t) = \sum_{t=1}^{\tau} r_n(\tilde{I}_t; x) \qquad (n = \nu(r_{i+1}), \ldots, \nu(r_{i+2})).$$

Offensichtlich bilden die Treppenfunktionen $\psi_n(x)$ $(n = \nu(r_0)+1, \ldots, \nu(r_{i+2}))$ ein K-beschränktes ONS in (0, 3).

Durch Rekursion werden wir zeigen folgendes: für jedes λ

$$(\lambda = 1, \ldots, \nu(r_i+1)-\nu(r_i))$$

gibt es eine Anordnung

$$\sum_{k=1}^{\nu(r_{i+1})-\nu(r_0)+2\lambda\bar{N}\nu(r_i+1)} c_{m(k,\lambda)}\psi_{m(k,\lambda)}(x) \tag{68}$$

der Summe

$$\sum_{n=\nu(r_0)+1}^{\nu(r_{i+1})+2\lambda\bar{N}\nu(r_i+1)} c_n\psi_n(x) \tag{69}$$

derart, daß mit gewissen Indizes $1 \leqq s'(\lambda, l) \leqq s''(\lambda, l) \leqq \nu(r_{i+1})+2\lambda\bar{N}\nu(r_i+1)$

$$\sum_{k=s'(l,m)}^{s''(\lambda,l)} c_{m(k,\lambda)}\psi_{m(k,\lambda)}(x) > \frac{C_{13}(K)}{32}\sum_{j=1}^{i+1} C_j \tag{70}$$

$$(x \in E_l(i+1);\ l = 1, \ldots, 2\lambda N\nu(r_i+1)),$$

und mit gewissen Indizes $1 \leqq \bar{s}'(\lambda, l) \leqq \bar{s}''(\lambda, l) \leqq \nu(r_{i+1})+2\lambda\bar{N}\nu(r_i+1)$

$$\sum_{k=\bar{s}'(\lambda,l)}^{\bar{s}''(\lambda,l)} c_{m(k,\lambda)}\psi_{m(k,\lambda)}(x) > \frac{C_{13}(K)}{32}\sum_{j=1}^{i} C_j \tag{71}$$

$$(x \in E_l(i);\ l = \lambda+1, \ldots, \nu(r_i+1)-\nu(r_i)).$$

Wir nehmen für die Summe $\sum_{n=\nu(r_0)+1}^{\nu(r_{i+1})+2\bar{N}\nu(r_i+1)} c_n\psi_n(x)$ die Anordnung

$$\sum_{k=1}^{\nu(r_{i+1})+2\bar{N}\nu(r_i+1)} c_{m(k,1)}\psi_{m(k,1)}(x) =$$

$$= c_{n(1,i)}\psi_{n(1,i)}(x)+\cdots+c_{n(i'(1,i)-1,i)}\psi_{n(i'(1,i)-1,i)}(x)+$$
$$+c_{n(1)}\psi_{n(1)}(x)+\cdots+c_{n(\bar{N}\nu(r_i+1))}\psi_{n(\bar{N}\nu(r_i+1))}(x)+$$
$$+c_{n(i'(1,i),i)}\psi_{n(i'(1,i),i)}(x)+\cdots+c_{n(i''(1,i),i)}\psi_{n(i''(1,i),i)}(x)+$$
$$+c_{n(\bar{N}\nu(r_i+1)+1)}\psi_{n(\bar{N}\nu(r_i+1)+1)}(x)+\cdots+c_{n(2\bar{N}\nu(r_i+1))}\psi_{n(2\bar{N}\nu(r_i+1))}(x)+$$
$$+c_{n(i''(1,i)+1,i)}\psi_{n(i''(1,i)+1,i)}(x)+\cdots+c_{n(\nu(r_{i+1})-\nu(r_0),i)}\psi_{n(\nu(r_{i+1})-\nu(r_0),i)}(x),$$

wobei die entsprechenden Anordnungen, bzw. die entsprechenden Indizes in (46) im Falle i, bzw. in (65) definiert sind. Aus (46) im Falle i, aus (66) und (67), weiterhin aus der Definition der Funktionen $\psi_n(x)$ ergibt sich, daß (70) und (71) im Falle $\lambda=1$ bestehen.

Es sei $\lambda(\geqq 1)$ eine ganze Zahl. Wir nehmen an, daß es eine Anordnung (68) der Summe (69) derart gibt, daß (70) und (71) für λ erfüllt sind.

Dann setzen wir für die Summe $\sum_{n=\nu(r_0)+1}^{\nu(r_{i+1})+2(\lambda+1)\bar{N}\nu(r_i+1)} c_n\psi_n(x)$ die Anordnung

$$\sum_{k=1}^{\nu(r_{i+1})+2(\lambda+1)\bar{N}\nu(r_i+1)} c_{m(k,\lambda+1)}\psi_{m(k,\lambda+1)}(x) =$$

$$= c_{m(1,\lambda)}\psi_{m(1,\lambda)}(x)+\cdots+c_{m(\bar{s}'(\lambda,\lambda+1)-1,\lambda)}\psi_{m(\bar{s}'(\lambda,\lambda+1)-1,\lambda)}(x)+$$
$$+c_{n(2\lambda\bar{N}\nu(r_i+1)+1)}\psi_{n(2\lambda\bar{N}\nu(r_i+1)+1)}(x)+\cdots$$
$$\cdots+c_{n(2\lambda\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1))}\psi_{n(2\lambda\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1))}(x)+$$
$$+c_{m(\bar{s}'(\lambda,\lambda+1),\lambda)}\psi_{m(\bar{s}'(\lambda,\lambda+1),\lambda)}(x)+\cdots+c_{m(\bar{s}''(\lambda,\lambda+1),\lambda)}\psi_{m(\bar{s}''(\lambda,\lambda+1),\lambda)}(x)+$$
$$+c_{n(2\lambda\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1)+1)}\psi_{n(2\lambda\bar{N}\nu(r_i+1)+\bar{N}\nu(r_i+1)+1)}(x)+\cdots$$
$$\cdots+c_{n(2(\lambda+1)\bar{N}\nu(r_i+1))}\psi_{n(2(\lambda+1)\bar{N}\nu(r_i+1))}(x)+$$
$$+c_{m(\bar{s}''(\lambda,\lambda+1)+1,\lambda)}\psi_{m(\bar{s}''(\lambda,\lambda+1)+1,\lambda)}(x)+\cdots$$
$$+c_{m(\nu(r_{i+1})+2(\lambda+1)\bar{N}\nu(r_i+1))}\psi_{m(\nu(r_{i+1})+2(\lambda+1)\bar{N}\nu(r_i+1))}(x),$$

wobei die entsprechenden Anordnungen, bzw. Indizes in (65), bzw. in (70) und (71) definiert sind. Aus (66), (67), (70), (71) und aus der Definition der Funktionen $\psi_n(x)$

ergibt sich, daß für diese Anordnung (70) und (71) im Falle $\lambda+1$ bestehen. Durch vollständiger Induktion bekommen wir also die Anordnung

$$\sum_{t=1}^{\nu(r_{i+1}+1)-\nu(r_0)} c_{n(t,i+1)}\psi_{n(t,i+1)}(x)$$

der Summe $\sum_{n=\nu(r_0)+1}^{\nu(r_{i+1}+1)} c_n\psi_n(x)$ derart, daß mit gewissen Indizes $1 \leqq i'(l,i+1) \leqq$ $\leqq i''(l,i+1) \leqq \nu(r_{i+1}+1)-\nu(r_0)$

$$\sum_{t=i'(l,i+1)}^{i''(l,i+1)} c_{n(t,i+1)}\psi_{n(t,i+1)}(x) > \frac{C_{13}(K)}{32}\sum_{j=1}^{i+1} C_j \qquad (x\in E_l(i+1))$$

$(l = 1, \ldots, \nu(r_{i+1}+1)-\nu(r_{i+1}))$ erfüllt ist. Wir setzen $n(t,i+1) = \nu(r_0)+t$ $(t = \nu(r_{i+1}+1)+1, \ldots, \nu(r_{i+2}))$. Damit bekommen wir eine Anordnung

$$\sum_{t=1}^{\nu(r_{i+2})-\nu(r_0)} c_{n(t,i+1)}\psi_{n(t,i+1)}(x)$$

der Summe $\sum_{n=\nu(r_0)+1}^{\nu(r_{i+2})} c_n\psi_n(x)$ derart, daß (46) im Falle $i+1$ besteht. Durch vollständiger Induktion erhalten wir also eine Anordnung $\sum_{k=1}^{\nu(k_2)-\nu(k_1)} c_{m(l)}\psi_{m(l)}(x)$ der Summe $\sum_{n=\nu(k_1)+1}^{\nu(k_2)} c_n\psi_n(x)$ derart, daß

$$\max_{\nu(k_1)<\lambda\leqq\nu(k_2)} \sum_{l=1}^{\lambda} c_{m(l)}\psi_{m(l)}(x) > \frac{C_{13}(K)}{64}\sum_{j=1}^{j_0} C_j \qquad (x\in\bar{E}) \tag{72}$$

besteht, wobei

$$\bar{E} = \bigcup_{l=1}^{\nu(r_{j_0}+1)-\nu(r_{j_0})} E_l(j_0)$$

ist. Nach (45) ist

$$m(\bar{E}) = C_{11}(K)/16. \tag{73}$$

Es sei $C_{14}(K)$ diejenige positive Konstante, für die

$$1/3C_{14}(K)+(1-1/C_{14}(K))K^2 = 1 \tag{74}$$

gilt $\left(C_{14}(K) = \dfrac{3K^2-1}{3K^2-3}\right)$. Wir setzen endlich

$$\chi_n(x) = \begin{cases} \psi_n(3C_{14}(K)x) & (x\in(0,1/C_{14}(K))), \\ Kr_n((x-1/C_{14}(K))/(1-1/C_{14}(K)) & (x\in(1/C_{14}(K),1)) \end{cases}$$

$(n = \nu(k_1)+1, \ldots, \nu(k_2))$. Nach (74) bilden diese Funktionen ein K-beschränktes ONS. Es sei $F(\subseteq(0,1))$ diejenige Menge, die aus $\bar{E}$ mit der linearen Transformation $y = x/3C_{14}(K)$ entsteht. Aus (74) folgt

$$m(F)=C_{15}(K). \tag{75}$$

Weiterhin aus (72) erhalten wir

$$\max_{\nu(k_1)<\lambda\leqq\nu(k_2)} \sum_{l=1}^{\lambda} c_{m(l)}\chi_{m(l)}(x) > C_{16}(K)\sum_{j=1}^{j_0} C_j \tag{76}$$

$$\left(x\in F;\ C_{16}(K) = \frac{C_{13}(K)}{64}\right).$$

Aus der Definition von c_n, weiterhin aus (43), (44), (75) und (76) folgt

$$I^{\frac{1}{2}}(c_{m(1)}, \ldots, c_{m(\nu(k_2)-\nu(k_1))}; K) > \tag{77}$$

$$> C_{17}(K)\sum_{k=k_1}^{k_2-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}}.$$

Da nach der Definition der Folge $\{c_n\}$ $|c_{m(l)}|\leqq|a_{m(l)}|$ $(l = 1, \ldots, \nu(k_2)-\nu(k_1))$ gilt, nach einem bekannten Resultate ([8], Hilfssatz II) ergibt sich

$$I^{\frac{1}{2}}(a_{m(1)}, \ldots, a_{m(\nu(k_2)-\nu(k_1))}; K) \geqq C_{18}(K) I^{\frac{1}{2}}(c_{m(1)}, \ldots, c_{m(\nu(k_2)-\nu(k_1))}; K). \tag{78}$$

Aus (77) und (78) bekommen wir

$$I^{\frac{1}{2}}(a_{m(1)}, \ldots, a_{m(\nu(k_2)-\nu(k_1))}; K) \geqq C_{19}(K)\sum_{k=k_1}^{k_2-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}}.$$

Daraus mit Anwendung des Hilfssatzes III erhalten wir den Hilfssatz I im Falle (42).

7. Endlich betrachten wir den Fall

(79)

$$\frac{1}{2}\sum_{k=k_1+1}^{k_2-1}\left(a^2_{\nu(k)+1}+\sum_{l=2}^{\nu(k+1)-\nu(k)} a^2_{\nu(k)+l}\log^2 l\right)^{\frac{1}{2}} > \sum_{k=k_1+1}^{k_2-1} |a_{\nu(k+1)}|\left(1+\sum_{l=2}^{\nu(k+1)-\nu(k)}\log^2 l\right)^{\frac{1}{2}}.$$

Wir setzen $c_n=a_n$ $(n = \nu(k_1)+1, \ldots, \nu(k_1+1)$, $c_n=a_{\nu(k)+p(\nu(k)-\nu(k-1))}$

$$(n = \nu(k)+(p-1)(\nu(k)-\nu(k-1))+1, \ldots, \nu(k)+p(\nu(k)-\nu(k-1));$$

$$p = 1, \ldots, (\nu(k+1)-\nu(k))(\nu(k)-\nu(k-1)),\quad k = k_1+1, \ldots, k_2-1).$$

Aus (79) folgt durch einfacher Rechnung

$$\sum_{k=k_1}^{k_2-1}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}} \leqq \left(c_{\nu(k_1)+1}^2+\sum_{l=2}^{\nu(k_1+1)-\nu(k_1)} c_{\nu(k_1)+l}^2\log^2 l\right)^{\frac{1}{2}}+$$

$$+\sum_{k=k_1+1}^{k_2-1}|a_{\nu(k)}|\left(1+\sum_{l=2}^{\nu(k)-\nu(k-1)}\log^2 l\right)^{\frac{1}{2}}+\sum_{k=k_1+1}^{k_2-1}\left(c_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} c_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}} \leqq$$

$$\leqq 2\left(c_{\nu(k_1)+1}^2+\sum_{l=2}^{\nu(k_1+1)-\nu(k_1)} c_{\nu(k_1)+l}^2\log^2 l\right)^{\frac{1}{2}}+\sum_{k=k_1+1}^{k_2-1}|a_{\nu(k+1)}|\left(1+\sum_{l=2}^{\nu(k+1)-\nu(k)}\log^2 l\right)^{\frac{1}{2}}+$$

$$+\sum_{k=k_1+1}^{k_2-1}\left(c_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} c_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}} \leqq$$

$$\leqq 2\sum_{k=k_1}^{k_2-1}\left(c_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} c_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}}+$$

$$+\frac{1}{2}\sum_{k=k_1}^{k_2-1}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}},$$

woraus folgt

$$\sum_{k=k_1}^{k_2-1}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}} \leqq 4\sum_{k=k_1}^{k_2-1}\left(c_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} c_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}}.$$

Dann gibt es eine ganze Zahl s $(0\leqq s\leqq 2)$ mit

$$\frac{1}{12}\sum_{k=k_1}^{k_2-1}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}} \leqq$$

$$\leqq \sum_{\substack{i \\ k_1\leqq 3i+s\leqq k_2-1}}\left(c_{\nu(3i+s)+1}^2+\sum_{l=2}^{\nu(3i+s+1)-\nu(3i+s)} c_{\nu(3i+s)+l}^2\log^2 l\right)^{\frac{1}{2}};$$

die Indizes $(k_1 \leqq)\, 3i+s\, (\leqq k_2-1)$ bezeichnen wir der Reihe nach mit $r_1<\cdots<r_{j_0}$; weiterhin sei $r_0=k_1$, $r_{j_0+1}=k_2$.

Dann können wir den Beweis in 6 mit dieser Folge $\{c_n\}$ und mit diesen Indizes r_j wiederholen, und so bekommen wir den Hilfssatz I auch im Falle (79).

Damit haben wir den Hilfssatz I vollständig bewiesen.

8. Endlich werden wir Satz I mit Anwendung des Hilfssatzes I beweisen. Wir nehmen $S=\infty$ an.

Ist $\{a_n\}\notin l^2$, dann divergiert die Rademachersche Reihe

$$\sum a_n r_n(x)$$

nach dem Khintchine—Kolmogoroffschen Satz in natürlicher Anordnung ihrer Glieder fast überall.

Wir nehmen endlich $S=\infty$ und $\{a_n\}\in l^2$ an. In diesem Falle können wir $|a_1|\geqq \geqq |a_2|\geqq\cdots$ annehmen. Also ist

$$\sum_{k=0}^{\infty}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}}\geqq A_4\sum_{k=0}^{\infty}\left\{\sum_{n=\nu(k)+1}^{\nu(k+1)} a_n^2\log^2 n\right\}^{\frac{1}{2}}=S=\infty.$$

Dann können wir eine Indexfolge $(C_0(K)\leqq)\varkappa_1<\cdots<\varkappa_r<\cdots$ derart angeben, daß

$$\sum_{k=\varkappa_r}^{\varkappa_{r+1}-1}\left(a_{\nu(k)+1}^2+\sum_{l=2}^{\nu(k+1)-\nu(k)} a_{\nu(k)+l}^2\log^2 l\right)^{\frac{1}{2}}\geqq 1\qquad (r=1,2,\ldots) \tag{80}$$

besteht.

Durch Induktion werden wir ein K-beschränktes ONS von Treppenfunktionen $\{\Phi_n(x)\}$ und eine Folge von einfachen Mengen $E_r(\subseteq(0,1))$ mit folgenden Eigenschaften definieren:

a) Die Mengen E_r $(r=1,2,\ldots)$ sind stochastich unabhängig, und für jedes r ist.

$$m(E_r)\geqq C_1(K). \tag{81}$$

b) Für jedes r gibt es eine Anordnung

$$\sum_{k=\nu(\varkappa_r)+1}^{\nu(\varkappa_{r+1})} a_{n(k)}\Phi_{n(k)}(x)$$

der Summe

$$\sum_{n=\nu(\varkappa_r)+1}^{\nu(\varkappa_{r+1})} a_n\Phi_n(x)$$

derart, daß

$$\max_{\nu(\varkappa_r)<i\leqq j\leqq\nu(\varkappa_{r+1})}\left|\sum_{k=i}^{j} a_{n(k)}\Phi_{n(k)}(x)\right|\geqq C_2(K)\qquad (x\in E_r) \tag{82}$$

gilt. ($C_0(K)$, $C_1(K)$ und $C_2(K)$ sind im Hilfssatz I definiert.)

Es sei

$$\Phi_n(x)=r_n(x)\qquad (n=1,\ldots,\nu(\varkappa_1)).$$

Wir wenden dann den Hilfssatz I im Falle $k_1=\varkappa_1$, $k_2=\varkappa_2$, $\{a_n\}$

$$(n=\nu(\varkappa_1)+1,\ldots,\nu(\varkappa_2))$$

an, die entsprechenden Funktionen, bzw. die entsprechenden Menge bezeichnen wir mit $\bar{\Phi}_n(x)$ $(n=\nu(\varkappa_1)+1,\ldots,\nu(\varkappa_2))$, bzw. mit $\bar{E}_1$. Da die Funktionen $\Phi_n(x)$ $(n=1,\ldots,\nu(\varkappa_1))$ Treppenfunktionen sind, gibt es eine Zerlegung von (0, 1) in endlich viele paarweise disjunkte Intervalle $I_1,\ldots,I_\sigma$ derart, daß jede Funktion $\Phi_n(x)$

$(n=1, \ldots, \nu(\varkappa_1))$ in jedem Intervall I_s konstant ist; die zwei Hälfte von I_s bezeichnen wir mit I_s', bzw. mit I_s''. Wir setzen

$$\Phi_n(x) = \sum_{s=1}^{\sigma} \bar{\Phi}_n(I_s'; x) - \sum_{s=1}^{\sigma} \bar{\Phi}_n(I_s''; x) \qquad (n = \nu(\varkappa_1)+1, \ldots, \nu(\varkappa_2))$$

und

$$E_1 = \bigcup_{s=1}^{\sigma} (\bar{E}_1(I_s') \cap \bar{E}_1(I_s'')).$$

Offensichtlich bilden die Treppenfunktionen $\Phi_n(x)$ $(n=1, \ldots, \nu(\varkappa_2))$ ein K-beschränktes ONS; die Menge E_1 ist einfach; weiterhin, auf Grund des Hilfssatzes I und (80) sind (81) und (82) für $r=1$ erfüllt.

Es sei $r_0(\geqq 2)$ eine natürliche Zahl. Wir nehmen an, daß die Treppenfunktionen $\Phi_n(x)$ $(n=1, \ldots, \nu(\varkappa_{r_0}))$ und die einfachen Mengen $E_1, \ldots, E_{r_0-1}$ schon derart definiert sind, daß diese Funktionen ein K-beschränktes ONS bilden, diese Mengen stochastisch unabhängig sind, weiterhin (81) und (82) für $r = 1, \ldots, r_0-1$ erfüllt sind.

Dann wenden wir den Hilfssatz I im Falle $k_1=\varkappa_{r_0}$, $k_2=\varkappa_{r_0+1}$, $\{a_n\}$ $(n = \nu(\varkappa_{r_0})+1, \ldots, \nu(\varkappa_{r_0+1}))$. Die entsprechenden Funktionen, bzw. die entsprechende Menge bezeichnen wir mit $\bar{\Phi}_n(x)$ $(n = \nu(\varkappa_{r_0})+1, \ldots, \nu(\varkappa_{r_0+1}))$, bzw. mit $\bar{E}_{r_0}$. Da die Funktionen $\Phi_n(x)$ $(n=1, \ldots, \nu(\varkappa_{r_0}))$ Treppenfunktionen und die Mengen $E_1, \ldots, E_{r_0-1}$ einfach sind, gibt es eine Zerlegung des Intervalls (0, 1) in endlich viele paarweise disjunkte Intervalle $J_1, \ldots, J_\varrho$ derart, daß jede Funktion $\Phi_n(x)$ $(n=1, \ldots, \nu(\varkappa_{r_0}))$ in jedem J_r konstant ist, und jede Menge E_r $(r = 1, \ldots, r_0-1)$ die Vereinigung gewisser J_r ist. Die zwei Hälfte von J_r bezeichnen wir mit J_r', bzw. mit J_r''. Wir setzen

$$\Phi_n(x) = \sum_{r=1}^{\varrho} \bar{\Phi}_n(J_r'; x) - \sum_{r=1}^{\varrho} \bar{\Phi}_n(J_r''; x) \qquad (n = \nu(\varkappa_{r_0})+1, \ldots, \nu(\varkappa_{r_0+1}))$$

und

$$E_{r_0} = \bigcup_{r=1}^{\varrho} (\bar{E}_{r_0}(J_r') \cup \bar{E}_{r_0}(J_r'')).$$

Offensichtlich bilden die Treppenfunktionen $\Phi_n(x)$ $(n=1, \ldots, \nu(\varkappa_{r_0+1}))$ ein K-beschränktes ONS, und die Mengen $E_1, \ldots, E_{r_0}$ sind stochastisch unabhängig. Weiterhin, auf Grund des Hilfssatzes I und (80) folgt, daß (81) und (82) auch für $r = r_0+1$ bestehen. Das angekündigte Funktionensystem $\{\Phi_n(x)\}$ und die Mengenfolge $\{E_r\}$ mit den erwähnten Eigenschaften ergibt sich also durch Induktion.

Wir betrachten die in *b)* angegebene Anordnung

$$\sum_{k=1}^{\infty} a_{n(k)} \Phi_{n(k)}(x) \qquad (\text{für } 1 \leqq k \leqq \nu(\varkappa_1) \text{ ist } n(k)=k) \tag{83}$$

der Reihe (3). Ist $x \in \overline{\lim\limits_{r\to\infty}} E_r$, so gilt (82) für unendlich viele r. Daraus folgt, daß die Reihe (3) im Punkt x divergiert. Wegen der stochastischen Unabhängigkeit der Mengenfolge $\{E_r\}$ und wegen (81) folgt

$$m(\overline{\lim_{r\to\infty}} E_r) = 1$$

durch Anwendung des zweiten Borel—Cantellischen Lemmas. Die Reihe (83) divergiert also fast überall.

Damit haben wir den Satz I bewiesen.

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On the strong approximation of orthogonal series

By L. LEINDLER in Szeged

Introduction

Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval (a, b). We consider the orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=1}^{\infty} c_n^2 < \infty. \tag{1}$$

By the Riesz—Fischer theorem, the series (1) converges in L^2 to a square-integrable function $f(x)$. Let us denote the partial sums and the (C, α)-means of the series (1) by $s_n(x)$ and $\sigma_n^\alpha(x)$, respectively.

In [4] and [5] we proved among others the following theorems.

Theorem I. *If* $0<\gamma<1$ *and*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty, \tag{2}$$

then

$$f(x)-\sigma_n^1(x) = o_x(n^{-\gamma})$$

almost everywhere in (a, b).

Theorem II. *If* (2) *is satisfied with some positive* γ *then*

$$f(x)-\frac{1}{n}\sum_{k=n}^{2n-1} s_k(x) = o_x(n^{-\gamma})$$

almost everywhere in (a, b).

G. SUNOUCHI [7] generalized Theorem I to strong approximation in the following way:

Theorem III. *If* (2) *holds with* $0<\gamma<1$, *then*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |f(x)-s_\nu(x)|^k\right\}^{1/k} = o_x(n^{-\gamma})$$

holds almost everywhere in (a, b) *for any* $\alpha>0$ *and* $0<k<\gamma^{-1}$, *where* $A_n^\alpha = \binom{n+\alpha}{n}$.

In the first part of the present paper we continue this generalization and prove the following theorems.

Theorem 1. *Suppose that* $0<\gamma<1$ *and* $0<k<\gamma^{-1}$ *and that*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty. \tag{3}$$

Suppose further that $\alpha(>0)$ *and* β *satisfy the inequality*

$$\min\left(1, \alpha, \frac{k}{2}\right) > (1-\beta)k. \tag{4}$$

Then we have

$$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |f(x)-\sigma_\nu^{\beta-1}(x)|^k\right\}^{1/k} = o_x(n^{-\gamma}) \tag{5}$$

almost everywhere in (a, b).

Theorem 2. *Suppose that* $0<k\leqq 2$ *and* $\gamma>0$, *and that* (3) *is satisfied. Then we have*

$$\left\{\frac{1}{n} \sum_{\nu=n}^{2n} |s_\nu(x)-f(x)|^k\right\}^{1/k} = o_x(n^{-\gamma}) \tag{6}$$

almost everywhere in (a, b).

It is easy to see that in the special case $\beta=1$ Theorem 1 reduces to Theorem III. In connection with very strong summability SUNOUCHI [6] proved

Theorem IV. *If*

$$\sum_{n=4}^{\infty} c_n^2 (\log\log n)^2 < \infty, \tag{7}$$

then

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_{k_\nu}(x)-f(x)|^k = 0$$

holds for any $\alpha>0$ *and* $k>0$, *almost everywhere in* (a, b), *for any increasing sequence* $\{k_\nu\}$.

In the special case $\alpha=1$ and $0<k\leqq 2$ this theorem was proved by TANDORI [8], and this special case was generalized by us ([3], Theorem 1) as follows:

Theorem V. *Under the condition* (7) *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n} |s_{l_\nu}(x)-f(x)|^k = 0$$

almost everywhere in (a, b) *for any* $0<k\leqq 2$ *and for any (not necessarily monotonic) seguence* $\{l_\nu\}$ *of distinct non-negative integers.*

Now we prove similar results for very strong approximation:

Theorem 3. *Suppose that* $0<\gamma<1$, $0<k<\gamma^{-1}$ *and* $\alpha>0$ *and that* (3) *is satisfied. Then we have*

$$\left\{\frac{1}{A_n^{(\alpha)}}\sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)}|s_{k_\nu}(x)-f(x)|^k\right\}^{1/k}=o_x(n^{-\gamma})$$

almost everywhere in (a, b) *for any increasing sequence* $\{k_\nu\}$.

Theorem 4. *If* $0<\gamma\leqq 1/2$, $0<k\leqq 2$, $k\gamma<1$ *and*

(8) $$\sum_{n=4}^{\infty} c_n^2 n^{2\gamma}(\log\log n)^2<\infty,$$

then

(9) $$\left\{\frac{1}{n}\sum_{\nu=1}^{n}|s_{l_\nu}(x)-f(x)|^k\right\}^{1/k}=o_x(n^{-\gamma})$$

almost everywhere in (a, b) *for any (not necessarily monotonic) sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

§ 1. Lemmas

The following lemmas will be required for the proofs of the theorems.

Lemma 1 ([1], p. 359). *Let* $r\geqq l>1$, $\bar{\gamma}>0$, $\bar{\alpha}>\bar{\gamma}-1$ *and* $\bar{\beta}\geqq\bar{\alpha}+l^{-1}-r^{-1}$. *Then*

$$\left\{\sum_{n=0}^{\infty}(n+1)^{r\bar{\gamma}-1}|\tau_n^{\bar{\beta}}(x)|^r\right\}^{1/r}\leqq K\left\{\sum_{n=0}^{\infty}(n+1)^{l\bar{\gamma}-1}|\tau_n^{\bar{\alpha}}(x)|^l\right\}^{1/l}, \quad {}^{*)}$$

where $\tau_n^\alpha(x)=\alpha(\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x))$.

Lemma 2 ([7], Lemma 1). *If*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma}<\infty \quad \textit{with} \quad 0<\gamma<1,$$

then

$$\int_a^b\left\{\sum_{n=0}^{\infty}(n+1)^{2\gamma-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)|^2\right\}dx\leqq K\sum_{n=1}^{\infty}c_n^2 n^{2\gamma}$$

for any $\alpha>\dfrac{1}{2}$.

*) $K, K_1, K_2, \ldots$ will denote positive constants not necesarily the same at each occurrence.

Lemma 3 ([2], p. 162). *Let* $\{\psi_k(x)\}$ $(k=1, 2, \ldots, N)$ *be an orthogonal system in* (a, b) *and let*

$$a_k^2 = \int_a^b \psi_k^2(x)\,dx \qquad (k=1, 2, \ldots, N).$$

Then there exists a function $\delta(x)$ *such that*

$$|\psi_1(x)+\cdots+\psi_i(x)| \leqq \delta(x) \qquad (i=1, 2, \ldots, N)$$

in (a, b) *and*

$$\int_a^b \delta^2(x)\,dx \leqq K \log^2 N \sum_{k=1}^{N} a_k^2.$$

Lemma 4. *Under the hypothesis of Theorem* 1 *we have the inequality*

$$\int_a^b \left\{ \sup_{0\leqq n<\infty} \left(\frac{n^{k\gamma}}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(x)-\sigma_\nu^\beta(x)|^k \right)^{1/k} \right\}^2 dx \leqq K \sum_{n=1}^{\infty} c_n^2 n^{2\gamma}.$$

Proof. By (4) we can give a positive number $q(>1)$ such that

$$\min\left(1, \alpha, \frac{k}{2}\right) > \frac{1}{q} > (1-\beta)k \tag{1.1}$$

holds. An easy computation using the inequalities (1.1) gives that

$$\beta > 1-\frac{1}{qk}, \quad p(1-\alpha) < 1 \quad \text{and} \quad qk>2,$$

where $p = \dfrac{q}{q-1}$. Applying Hölder's inequality with the above defined numbers p and q and using $p(1-\alpha) < 1$ we obtain that

$$\begin{aligned}(1.2)\qquad &\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\tau_\nu^\beta(x)|^k \leqq \\ &\leqq K \left\{ \sum_{\nu=0}^{n} (\nu+1)^{\gamma qk-1} |\tau_\nu^\beta(x)|^{qk} \right\}^{1/q} \left\{ \sum_{\nu=0}^{n} (A_{n-\nu}^{\alpha-1})^p (\nu+1)^{\frac{p}{q}-\gamma kp} \right\}^{1/p} \leqq \\ &\leqq K_1 \left\{ \sum_{\nu=0}^{n} (\nu+1)^{\gamma qk-1} |\tau_\nu^\beta(x)|^{qk} \right\}^{1/q} n^{\alpha-k\gamma}. \qquad ^{*)}\end{aligned}$$

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*) $$\begin{aligned}&\sum_{\nu=0}^{n} (A_{n-\nu}^{\alpha-1})^p (\nu+1)^{\frac{p}{q}-\gamma kp} \leqq \\ &\leqq \left(\sum_{\nu=0}^{n/2} + \sum_{\nu=n/2}^{n} \right) \leqq K \sum_{\nu=0}^{n/2} n^{(\alpha-1)p} (\nu+1)^{\frac{p}{q}-\gamma kp} + K \sum_{k=1}^{n/2} k^{(\alpha-1)p} n^{\frac{p}{q}-\gamma kp} \leqq \\ &\leqq K_1 (n^{(\alpha-1)p+(1-\gamma k)p} + n^{(\alpha-1)p+1+\frac{p}{q}-\gamma kp}) \leqq K_2 n^{(\alpha-\gamma k)p}.\end{aligned}$$

Since $\beta > 1-\frac{1}{qk}$ and $qk>2$ we can find α^* such that both inequalities

$$\alpha^* > \frac{1}{2} \quad \text{and} \quad \beta > \alpha^* + \frac{1}{2} - \frac{1}{qk} \tag{1.3}$$

hold. By (1.3) and $qk>2$ the conditions of Lemma 1 are fulfilled with $r=qk$, $l=2$, $\bar\gamma=\gamma$, $\bar\alpha=\alpha^*$ and $\bar\beta=\beta$. Applying Lemma 1 we obtain that

$$\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{\gamma qk-1}|\tau_n^{\beta}(x)|^{qk}\right\}^{1/qk} \leqq K\left\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_n^{\alpha^*}(x)|^2\right\}^{1/2}. \tag{1.4}$$

By (1.2) and (1.4) we have

$$\int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{k\gamma}}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}|\tau_\nu^{\beta}(x)|^k\right)^{1/k}\right\}^2 dx \leqq \tag{1.5}$$

$$\leqq K_1\int_a^b\left\{\sum_{\nu=0}^{n}(\nu+1)^{2\gamma-1}|\tau_n^{\alpha^*}(x)|^2\right\}dx.$$

Since $\alpha^* > \frac{1}{2}$ and $0<\gamma<1$, by (1.5) and Lemma 2, we have

$$\int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{k\gamma}}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}|\tau_\nu^{\beta}(x)|^k\right)^{1/k}\right\}^2 dx \leqq K_2\sum_{n=1}^{\infty}c_n^2 n^{2\gamma},$$

which is the statement of Lemma 4.

Lemma 5. *Let $\{\lambda_n\}$ be a monotonic sequence of positive numbers such that*

$$\sum_{n=1}^{m}\lambda_{2^n}^2 \leqq K\lambda_{2^m}^2. \tag{1.6}$$

If

$$\sum_{n=0}^{\infty}c_n^2\lambda_n^2<\infty,$$

then we have

$$s_{2^n}(x)-f(x) = o_x(\lambda_{2^n}^{-1}) \tag{1.7}$$

almost everywhere in (a, b).

Proof. An easy computation gives (1.7). In particular

$$\sum_{n=0}^{\infty}\int_a^b\lambda_{2^n}^2|s_{2^n}(x)-f(x)|^2\,dx = \sum_{n=0}^{\infty}\lambda_{2^n}^2\sum_{m=n}^{\infty}\sum_{i=2^m+1}^{2^{m+1}}c_i^2 =$$

$$=\sum_{m=0}^{\infty}\left(\sum_{i=2^m+1}^{2^{m+1}}c_i^2\right)\sum_{n=0}^{m}\lambda_{2^n}^2 \leqq K\sum_{m=0}^{\infty}\lambda_{2^m}^2\sum_{i=2^m+1}^{2^{m+1}}c_i^2<\infty.$$

Lemma 6. *Let $\{\lambda_n\}$ be a monotonic sequence of positive numbers with* (1. 6). *If*

$$\sum_{n=0}^{\infty} c_n^2 \lambda_{2n}^2 < \infty,$$

then we have

$$\left\{\frac{1}{n}\sum_{\nu=n}^{2n} |s_\nu(x)-f(x)|^k\right\}^{1/k} = o_x(\lambda_n^{-1}) \tag{1. 8}$$

almost everywhere in (a, b) *for any* $0<k\leqq 2$.

Proof. By an easy computation we obtain that

$$\sum_{m=0}^{\infty} \int_a^b \frac{\lambda_{2^{m+1}}^2}{2^m} \sum_{k=2^m+1}^{2^{m+1}} |s_k(x)-f(x)|^2\,dx \leqq$$

$$\leqq \sum_{m=0}^{\infty} \lambda_{2^{m+1}}^2 \sum_{i=2^m+1}^{\infty} c_i^2 \leqq K \sum_{m=0}^{\infty} \lambda_{2^{m+1}}^2 \sum_{i=2^m+1}^{2^{m+1}} c_i^2 \leqq K \sum_{n=0}^{\infty} c_n^2 \lambda_{2n}^2.$$

Hence

$$\lim_{m\to\infty} \frac{\lambda_{2^{m+1}}^2}{2^m} \sum_{\nu=2^m+1}^{2^{m+1}} |s_\nu(x)-f(x)|^2 = 0$$

follows almost everywhere in (a, b), which implies (1. 8) for $k=2$. Hence the statement (1. 8) for $0<k<2$ can be deduced by Hölder's inequality.

§ 2. Proof of the theorems

(Theorem 1.) Theorem III with $k=1$ implies that

$$\sigma_n^\beta(x)-f(x) = o_x(n^{-\gamma}).$$

In view of this we have

$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^\beta(x)-f(x)|^k = o_x(n^{-\gamma k}),$$

thus

$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |f(x)-\sigma_\nu^{\beta-1}(x)|^k \leqq \frac{K}{A_n^\alpha}\sum_{\nu=0}^{n} A_{u-\nu}^{\alpha-1} |\sigma_\nu^\beta(x)-\sigma_\nu^{\beta-1}(x)|^k + o_x(n^{-\gamma k}).$$

Next we show that the last sum is of order $o_x(n^{-\gamma k})$. For any positive ε, we choose N such that

$$\sum_{n=N/4}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3. \tag{2. 1}$$

Now we consider the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \quad \text{with} \quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq N, \\ 0 & \text{for} \quad n > N, \end{cases} \tag{2. 2}$$

and

$$\sum_{n=1}^{\infty} b_n \varphi_n(x) \quad \text{with} \quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq N, \\ c_n & \text{for} \quad n > N. \end{cases} \tag{2. 3}$$

Let us denote by $\sigma_n^\beta(a; x)$ and $\sigma_n^\beta(b; x)$, respectively, the n-th Cesàro means of order β of the series (2. 2) and (2. 3). It is clear that $\sigma_n^\beta(x) = \sigma_n^\beta(a; x) + \sigma_n^\beta(b; x)$. Since the number of the coefficients $a_n \neq 0$ is finite and $\gamma k < 1$ the sums

$$\frac{n^{\gamma k}}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(a; x) - \sigma_\nu^\beta(a; x)|^k \tag{2. 4}$$

converge clearly to zero almost everywhere in (a, b). Using Lemma 4 and (2. 1) we obtain

$$\int_a^b \left\{ \sup_{0 \leqq n < \infty} \left(\frac{n^{k\gamma}}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(b; x) - \sigma_\nu^\beta(b; x)|^k \right)^{1/k} \right\}^2 dx \leqq K_1 \varepsilon^3.$$

Hence

$$\operatorname{meas} \left\{ x \,\middle|\, \limsup \left(\frac{n^{k\gamma}}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(b; x) - \sigma_\nu^\beta(b; x)|^k \right)^{1/k} > \varepsilon \right\} \leqq K_1 \varepsilon,$$

this and (2. 4) imply that the sums

$$\frac{n^{k\gamma}}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(x) - \sigma_\nu^\beta(x)|^k$$

converge to zero almost everywhere in (a, b).

Collecting our results we obtain the statement of Theorem 1.

(Theorem 2.) Applying Lemma 6 with $\lambda_n = n^\gamma$ $(\gamma > 0)$ we obtain (6) immediately.

(Theorem 3.) We set

$$C_n = \left(\sum_{i=k_{n-1}+1}^{k_n} c_i^2 \right)^{1/2}$$

and

$$\Phi_n(x) = \begin{cases} C_n^{-1} \displaystyle\sum_{i=k_{n-1}+1}^{k_n} c_i \varphi_i(x) & \text{for} \quad C_n \neq 0, \\ (k_n - k_{n-1})^{-1/2} \displaystyle\sum_{i=k_{n-1}+1}^{k_n} \varphi_i(x) & \text{for} \quad C_n = 0. \end{cases}$$

The system $\{\Phi_n(x)\}$ is also an orthonormal one and the series $\Sigma C_n \Phi_n(x)$ satisfies obviously

$$\sum_{n=4}^{\infty} C_n^2 n^{2\gamma} < \infty.$$

Since

$$\sum_{k=0}^{n} C_k \Phi_k(x) = s_{k_n}(x),$$

applying Theorem 1 to the series $\Sigma C_n \Phi_n(x)$, we get the statement of Theorem 3.

(Theorem 4.) Under the condition (8) we have by Lemma 5

$$s_{2^n}(x) - f(x) = o_x(2^{-n\gamma}), \tag{2.5}$$

almost everywhere in (a, b). We set $C_m^2 = \sum_{n=2^m+1}^{2^{m+1}} c_n^2$.

Now we define sequences of indices $\{\mu_i(m)\}$ for every m. We put $\mu_0(m) = 2^m$. If $C_m \neq 0$ then let $\mu_i(m)$ $(1 \leqq i \leqq N_m)$ be the smallest natural number for which both inequalities

$$\sum_{n=\mu_{i-1}(m)+1}^{\mu_i(m)} c_n^2 \geqq \frac{C_m^2}{m} \quad \text{and} \quad \mu_i(m) \leqq 2^{m+1}$$

hold. It is clear that $N_m \leqq m$. If $C_m = 0$ then we set $\mu_1(m) = 2^{m+1}$. Applying Lemma 3 to the functions

$$\psi_i^{(m)}(x) = s_{\mu_i(m)}(x) - s_{\mu_{i-1}(m)}(x) \qquad (1 \leqq i \leqq N_m)$$

we have a function $\delta_m(x)$ such that

$$|s_{\mu_i(m)}(x) - s_{2^m}(x)| = \left|\sum_{j=1}^{i} \psi_j^{(m)}(x)\right| \leqq \delta_m(x) \qquad (1 \leqq i \leqq N_m) \tag{2.6}$$

in (a, b) and

$$\int_a^b \delta_m^2(x)\,dx \leqq K \log^2 N_m \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \leqq K_1 \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \log\log^2 n. \tag{2.7}$$

Then, by (8) and (2. 7), we have that

$$\sum_{m=0}^{\infty} 2^{2m\gamma} \int_a^b \delta_m^2(x)\,dx < \infty,$$

which implies, by (2. 6), that

$$2^{m\gamma} |s_{\mu_i(m)}(x) - s_{2^m}(x)| \to 0$$

almost everywhere in (a, b). Hence, by (2, 5),

$$|s_{\mu_i(m)}(x)-f(x)| = o_x(2^{-m\gamma}) \tag{2.8}$$

holds almost everywhere in (a, b).

Now we define a new sequence of indices $\{\mu_\nu\}$. If $\mu_i(m)\leqq l_\nu<\mu_{i+1}(m)$ then let $\mu_\nu=\mu_i(m)$, and if $\mu_{N_m}(m) \leqq l_\nu < \mu_0(m+1)$ then let $\mu_\nu=\mu_{N_m}(m)$.

Then, by (2. 8) and $k\gamma<1$, we have

$$\frac{1}{n}\sum_{\nu=1}^{n}|s_{l_\nu}(x)-f(x)|^k \leqq \frac{K}{n}\sum_{\nu=1}^{n}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k+\frac{K}{n}\sum_{\nu=1}^{n}|s_{\mu_\nu}(x)-f(x)|^k \leqq$$

$$\leqq \frac{K}{n}\sum_{\nu=1}^{n}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k+\frac{K}{n}\sum_{m=0}^{\log n}2^m o_x(2^{-m\gamma k}) \leqq \frac{K}{n}\sum_{\nu=1}^{n}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k+o_x(n^{-\gamma k}). \text{ *)}$$

From this point on the proof runs similarly to the proof of Theorem 1. Let us define N, $\{a_n\}$ and $\{b_n\}$ in the same way as under (2. 1), (2. 2) and (2. 3); furthermore let $s_n(a; x)$ and $s_n(b; x)$ denote the nth partial sums of series (2. 2) and (2. 3).

Since $s_n(x) = s_n(a; x)+s_n(b; x)$ we obtain

$$n^{\gamma k-1}\sum_{\nu=1}^{n}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k \leqq Kn^{\gamma k-1}\sum_{\nu=1}^{n}(|s_{l_\nu}(a;x)-s_{\mu_\nu}(a;x)|^k+|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k).$$

By $\gamma k<1$ the first sums converge to zero almost everywhere in (a,b). To estimate the second sums we use Hölder's inequality and obtain

$$n^{\gamma k-1}\sum_{\nu=1}^{n}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k \leqq$$

$$\leqq n^{\gamma k-1}\left\{\sum_{\nu=1}^{n}\nu^{2\gamma-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\right\}^{k/2}\left\{\sum_{\nu=1}^{n}\nu^{(1-2\gamma)\frac{k}{2-k}}\right\}^{1-k/2} \leqq$$

$$\leqq K\left\{\sum_{\nu=1}^{n}\nu^{2\gamma-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\right\}^{k/2}.$$

Hence

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(n^{\gamma k-1}\sum_{\nu=1}^{n}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\right)^{1/k}\right\}^2 dx \leqq \tag{2.9}$$

$$\leqq K\int_a^b\left(\sum_{\nu=1}^{\infty}\nu^{2\gamma-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\right)dx.$$

*) The logarithm used is with basis 2.

A standard computation gives that*)

$$\sum_{\nu=1}^{\infty} \nu^{2\gamma-1} \int_a^b |s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\,dx =$$

$$= \sum_{\nu=1}^{\infty} \nu^{2\gamma-1} \sum_{k=\mu_\nu+1}^{l_\nu} b_k^2 = \sum_{m=0}^{\infty} \sum_{2^m \leqq \mu_\nu < 2^{m+1}}^{(\nu)} \nu^{2\gamma-1} \sum_{k=\mu_\nu+1}^{l_\nu} b_k^2 \leqq$$

$$\leqq \sum_{m=[\log N]}^{\infty} \left(\sum_{2^m \leqq \mu_\nu < 2^{m+1}} \nu^{2\gamma-1} \right) \frac{C_m^2}{m} \leqq \sum_{m=[\log N]}^{\infty} \left(\sum_{\nu=1}^{2^m} \nu^{2\gamma-1} \right) C_m^2 \leqq K \sum_{k=N/2}^{\infty} c_k^2 k^{2\gamma}.$$

This and (2. 9) imply that

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(n^{\gamma k-1} \sum_{\nu=1}^{n} |s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k \right)^{1/k} \right\}^2 dx \leqq K \sum_{k=N/2}^{\infty} c_k^2 k^{2\gamma}.$$

Hence

$$\operatorname{meas}\left\{ x \,|\, \limsup \left(n^{\gamma k-1} \sum_{\nu=1}^{n} |s_{l_\nu}(b;x)-s_{\mu}(b;x)|^k \right)^{1/k} > \varepsilon \right\} \leqq K\varepsilon.$$

Thus we obtain that

$$\left(n^{\gamma k-1} \sum_{\nu=1}^{n} |s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k \right)^{1/k} = o_x(1)$$

holds almost everywhere in (a, b).

Collecting our results we obtain (9) almost everywhere in (a, b) and the theorem is proved.

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*) $\Sigma^{(\nu)}$ denotes that the sum is taken for ν. $[\alpha]$ denotes the integer part of α.

On generalized absolute Cesàro summability of orthogonal series

By ISTVÁN SZALAY in Szeged

As usual we denote by $\sigma_n^{(\alpha)}$ the nth Cesàro means of order α of a series Σa_n. The following definition is due to FLETT [1]: A series Σa_n is said to be $|C, \alpha, \gamma|_\varkappa$ summable, where $\varkappa \geqq 1$ and $\alpha > -1$, if the series $\Sigma n^{\varkappa\gamma+\varkappa-1}|\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^\varkappa$ is convergent.

We prove the following theorems:

Theorem 1. *Let* $\alpha > \frac{1}{2}$, $0 \leqq \gamma < 1$, $1 \leqq \varkappa \leqq 2$. *The condition*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{2\gamma} a_n^2 \right\}^{\varkappa/2} < \infty \tag{1}$$

is necessary and suffcient that for any orthonormal system $\{\varphi_n(x)\}$ *on* $(0, 1)$ *the series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{2}$$

be summable $|C, \alpha, \gamma|_\varkappa$ *almost everywhere in* $(0, 1)$.

This theorem reduces for $\alpha > \frac{1}{2}$, $\gamma = 0$ and $\varkappa = 1$ to a theorem of LEINDLER [2] which in turn contains a theorem of TANDORI [3], case $\alpha = 1$, $\gamma = 0$, $\varkappa = 1$.

The sequence of ideas of our proof is similar to that of LEINDLER.

Theorem 2. *Let* $0 \leqq \gamma < 1$ *and* $1 \leqq \varkappa \leqq 2$. *Then the conditions*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{2\gamma} a_n^2 \log n \right\}^{\varkappa/2} < \infty \quad \left(\text{for } \alpha = \frac{1}{2} \right)$$

and

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{1+2\gamma-2\alpha} a_n^2 \right\}^{\varkappa/2} < \infty \quad \left(\text{for } -1 < \alpha < \frac{1}{2} \right)$$

are sufficient that the series (2) *be summable* $|C, \alpha, \gamma|_\varkappa$ *for any orthonormal system* $\{\varphi_n(x)\}$ *in* $(0, 1)$, *almost everywhere in* $(0, 1)$.

In the special case $\gamma = 0$ $\varkappa = 1$ this theorem was proved by LEINDLER ([2], p. 253). The proof is similar to the proof of Theorem 1, so we omit it.

It is of some interest to remark the following corollary to Theorems 1 and 2 and to a theorem of FLETT (see [1], p. 359).

Corollary. *Let* $0 \leqq \gamma < 1$ *and* $\varkappa \geqq 2$. *The series* (2) *is* $|C, \alpha, \gamma|_\varkappa$ *summable for any orthonormal system* $\{\varphi_n(x)\}$ *almost everywhere in* (0, 1) *in each of the following three cases:*

$$\text{(i)} \quad \alpha > 1 - \frac{1}{\varkappa} \quad \textit{and} \quad \sum_{n=0}^{\infty} n^{2\gamma} a_n^2 < \infty,$$

$$\text{(ii)} \quad \alpha = 1 - \frac{1}{\varkappa} \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{2\gamma} a_n^2 \log n < \infty,$$

$$\text{(iii)} \quad \alpha \geqq \beta + \frac{1}{2} - \frac{1}{\varkappa} \quad \left(-1 < \beta < \frac{1}{2}\right) \quad \textit{and}$$

$$\sum_{n=0}^{\infty} n^{1+2\gamma-2\beta} a_n^2 < \infty, \quad 0 \leqq \gamma < \min(1, 1+\beta).$$

Proof of Theorem 1. Let $A_m^{(\alpha)} = \binom{m+\alpha}{m}$. Then we have:

$$0 < c_1(\alpha) \leqq \frac{A_m^{(\alpha)}}{m^\alpha} \leqq c_2(\alpha) \qquad (m > 0,\ \alpha > -1), \tag{3}$$

$$A_m^{(\alpha)} > 0 \qquad (m \geqq 0,\ \alpha > -1), \tag{4}$$

and

$$A_{m+1}^{(\alpha)} > A_m^{(\alpha)} \qquad (m \geqq 0,\ \alpha > 0), \tag{5}$$

where $c_1(\alpha)$ and $c_2(\alpha)$ are independent of m. (See ZYGMUND [4], p. 77.) We define

$$L_{n,\nu}^{(\alpha)} = \frac{A_{n+1-\nu}^{(\alpha)}}{A_{n+1}^{(\alpha)}} - \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} = \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} \cdot \frac{\nu\alpha}{(n+1-\nu)(n+1+\alpha)}.$$

From (3), (4) and (5) it easily follows that for any $n = 1, 2, \ldots;\ \nu = 0, 1, \ldots, n;\ \alpha > -1$, $\alpha \neq 0$:

$$0 < d_1(\alpha) \frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \leqq |L_{n,\nu}^{(\alpha)}| \leqq d_2(\alpha) \frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \tag{6}$$

and

$$\operatorname{sgn} L_{n,\nu}^{(\alpha)} = \operatorname{sgn} \alpha,$$

where $d_1(\alpha)$ and $d(\alpha)$ are independent of n.

First we prove the necessity of condition (1). Without loss of generality we may assume that $a_0 = a_1 = 0$ and $a_n \neq 0$ for $n \geqq 2$. We define by induction a special orthonormal system of step functions $\{\chi_n(x)\}$ $(n = 0, 1, \ldots)$ in (0, 1). Let

$$\chi_n(x) = r_n(x) \qquad (n = 0, 1, 2).^{1)}$$

[1]) $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ the n-th Rademacher function.

Let $s(\geqq 1)$ be any natural number. Suppose that the step functions $\chi_n(x)$ $(n=0, 1, \ldots, 2^s)$ have been defined such that $\{\chi_n(x)\}$ $(n=0, \ldots, 2^s)$ is a H-type system i.e. $\chi_n(x)\chi_m(x)=0$ for any $x\in(0, 1)$, if $2^k<n, m\leqq 2^{k+1}$ $n\neq m$ and $k = 0, 1, \ldots$ $\ldots, s-1$. Then the interval $(0, 1)$ can be dissected into subintervals J_ϱ $(1\leqq\varrho\leqq\varrho_s)$ such that on any J_ϱ every $\chi_n(x)$ $(n=0, 1, \ldots, 2^s)$ is constant. We define the following sequence:

$$\varrho_0^{(m)} = 0 \quad \text{and} \quad \varrho_k^{(m)} = \frac{1}{A_m^{2/\varkappa}} \sum_{n=1}^{k} a_{2^m+n}^2 \qquad (k=1, \ldots, 2^m),$$

where $A_m = \left\{ \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right\}^{\varkappa/2}$ $(m=0, 1, \ldots)$. Now we dissect every interval $J_\varrho=(u_\varrho, v_\varrho)$ into 2^s intervals as follows:

$$I_k(s, J_\varrho) = (u_\varrho^{(k)}, v_\varrho^{(k)}),$$

where

$$u_\varrho^{(k)} = u_\varrho+\mu(J_\varrho)\varrho_{k-1}^{(s)} \quad \text{and} \quad v_\varrho^{(k)} = u_\varrho+\mu(J_\varrho)\varrho_k^{(s)} \qquad (k=1, \ldots, 2^s).\ ^{2)}$$

Then we define

$$\chi_{2^s+k}(x) = \frac{A_s^{1/\varkappa}}{a_{2^s+k}} \sum_{\varrho=1}^{\varrho_s} r_s(x; I_k(s, J_\varrho)).^{3)}$$

These functions $\chi_n(x)$ $(2^s\leqq n\leqq 2^{s+1})$ are step functions and

$$\int_0^1 \chi_{2^s+k}^2(x)\,dx = \frac{A_s^{2/\varkappa}}{a_{2^s+k}^2} \sum_{\varrho=1}^{\varrho_s} \int_0^1 r_s^2(x; I_k(s, J_\varrho))\,dx =$$

$$= \frac{A_s^{2/\varkappa}}{a_{2^s+k}^2} \sum_{\varrho=1}^{\varrho_s} \frac{a_{2^s+k}^2}{A_s^{2/\varkappa}} \mu(J_\varrho) = \sum_{\varrho=1}^{\varrho_s} \mu(J_\varrho) = 1.$$

From the definition it is clear that the functions $\chi_n(x)$ $(n=0, 1, \ldots, 2^{s+1})$ give rise to an orthonormal system on $(0, 1)$ and for every $x\in(0, 1)$ we have

$$\chi_n(x)\chi_m(x) = 0 \qquad (2^l<n, m\leqq 2^{l+1};\ 0\leqq l\leqq s).$$

Hence, by induction, we get an infinite H-type system.

---

2) μ denotes Lebesgue measure.

3) If $I(u, v)$ is a finite interval and $h(x)$ is a function defined on $(0, 1)$, then

$$h(x; I) = \begin{cases} h\left(\dfrac{x-u}{v-u}\right), & \text{if} \quad u<x<v \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\int_u^v h(x; I)\,dx = \mu(I)\int_0^1 h(x)\,dx$.

We consider the series

$$\sum_{n=0}^{\infty} a_n \chi_n(x). \tag{7}$$

Denote $\bar{\sigma}_n^{(\alpha)}$ the n-th (C, α) means of series (7). Let us assume that the series (2) for any orthonormal system is $|C, \alpha, \gamma|_\varkappa$ summable almost everywhere in $(0, 1)$. Then we have

$$\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha)}(x)|^\varkappa < \infty$$

almost everywhere in $(0, 1)$.

Let $\varepsilon = \min\{1; 2^{-(7+3\varkappa+3\varkappa\alpha)} d_1^\varkappa(\alpha) d_2^\varkappa(\alpha)\}$, where $d_1(\alpha)$ and $d_2(\alpha)$ are the same constants as in (6). By the Egorov theorem there exists a measurable set E with $\mu(E) \geqq 1-\varepsilon$ and a positive constant K such that for every $x \in E$

$$\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha)}(x)|^\varkappa < K.$$

Hence

$$\sum_{n=2}^{\infty} \int_E n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha-1)}(x)|^\varkappa \leqq K\mu(E).$$

Let m and n be integers such that $2^m < n \leqq 2^{m+1}$. Then we put

$$R_l(x; m, n) = \sum_{\nu=2^l+1}^{2^{l+1}} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x) \qquad (l=0, 1, \ldots, m-1),$$

$$R_m(x; m, n) = \sum_{\nu=2^m+1}^{n} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x), \quad R_{m+1}(x; m, n) = \frac{1}{A_{n+1}^{(\alpha)}} a_{n+1} \chi_{n+1}(x).$$

These functions $R_l(x; m, n)$ $(l = 0, 1, \ldots, m+1)$ satisfy the conditions of the following

Lemma. (LEINDLER [2]) *Let $\{R_n(x)\}$ $(n=1, 2, \ldots)$ be a system of step functions defined on $(0, 1)$. Denote $J_s(n)$ $(n=1, 2, \ldots;\ s=1, 2, \ldots, s_n)$ the intervals on which $R_n(x)$ is constant. If for every $m > n$*

$$\int_{J_s(n)} \operatorname{sign} R_m(x)\, dx = 0 \qquad (s=1, \ldots, s_n),$$

then for any sequence of numbers $d_1, \ldots, d_N$ there exists a set E_k of subintervals such that for any $x \in E_k$

$$\left|\sum_{l=1}^{N} d_l R_l(x)\right| \geqq |d_{N-k} R_{N-k}(x)| \qquad (k=0, 1, \ldots, N-1)$$

and

$$\mu(E_k \cap J_s(N-k-1)) = \frac{\mu(J_s(N-k-1))}{2^{k+1}}$$

$$(k=0, 1, \dots, N-1;\ s=1, 2, \dots, s_{N-k-1};\ J_1(0)=(0, 1)).$$

We use this lemma in case $N = m+1$ and $k=3$. The suitable set E_k will be denoted by $E_3(m, n)$. Then we have:

(8)
$$\sum_{n=2^3+1}^{\infty} \int_E n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha)}(x)|^{\varkappa}\, dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \int_E \left| \sum_{\nu=0}^{n} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x) + \frac{1}{A_{n+1}^{(\alpha)}} a_{n+1}\chi_{n+1}(x)\right|^{\varkappa} dx =$$

$$= \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \int_E \left| \sum_{l=0}^{m+1} R_l(x; m, n)\right|^{\varkappa} dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \int_{E\cap E_3(m,n)} |R_{m-2}(x; m, n)|^{\varkappa}\, dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \left(\int_{E_3(m,n)} - \int_{E_3(m,n)-E_3(m,n)\cap E} \right) |R_{m-2}(x; m, n)|^{\varkappa}\, dx = S.$$

By the lemma we have:

$$\int_{E_3(m,n)} |R_{m-2}(x; m, n)|^{\varkappa}\, dx = \int_{E_3(m,n)} \left| \sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x)\right|^{\varkappa} dx =$$

$$= \int_{E_3(m,n)} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^{\varkappa} a_\nu^{\varkappa} |\chi_\nu(x)|^{\varkappa}\, dx \geqq$$

$$\geqq \sum_{k=1}^{2^{m-2}} (L_{n,2^{m-2}+k}^{(\alpha)})^{\varkappa} a_{2^{m-2}+k}^{\varkappa} \sum_{\varrho=1}^{\varrho_{m-2}} \int_{E_3(m,n)\cap I_k(m-2,J_\varrho)} \frac{A_{m-2}}{a_{2^{m-2}+k}^{\varkappa}}\, dx \geqq$$

$$\geqq \sum_{k=1}^{2^{m-2}} (L_{n,2^{m-2}+k}^{(\alpha)})^{\varkappa} A_{m-2} \sum_{\varrho-1}^{\varrho_{m-2}} \frac{\mu(I_k(m-2, J_\varrho))}{2^4} =$$

$$= \frac{1}{2^4} \sum_{k=1}^{2^{m-2}} (L_{n,2^{m-2}+k}^{(\alpha)})^{\varkappa} A_{m-2} \sum_{\varrho-1}^{\varrho_{m-2}} \frac{a_{2^{m-2}+k}^2}{A_{m-2}^{2/\varkappa}} \mu(J_\varrho) =$$

$$= \frac{1}{2^4} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^{\varkappa} A_{m-2}^{1-2/\varkappa} a_\nu^2.$$

In order to estimate the second integral in (8) we apply the Hölder inequality:

$$\int\limits_{E_3(m,n)-E_3(m,n)\cap E} |R_{m-2}(x;m,n)|^{\varkappa}\,dx \leqq \varepsilon^{\frac{2-\varkappa}{2}} \left(\int\limits_0^1 \left|\sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x)\right|^2 dx\right)^{\varkappa/2} =$$

$$= \varepsilon^{\frac{2-\varkappa}{2}} \left(\sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^2 a_\nu^2\right)^{\varkappa/2}.$$

Thus, by a standard computation, we obtain that

$$S \geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1}\cdot$$

$$\cdot\left\{2^{-4} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^{\varkappa} A_{m-2}^{1-\frac{2}{\varkappa}} a_\nu^2 - \varepsilon^{\frac{2-\varkappa}{2}} \left(\sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)})^2 a_\nu^2\right)^{\varkappa/2}\right\} \geqq$$

$$\geqq \sum_{m=3}^{\infty} 2^{-4} d_1^{\varkappa}(\alpha) A_{m-2}^{1-\frac{2}{\varkappa}} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} n^{\varkappa\gamma+\varkappa-1} \left(\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}}\right)^{\varkappa} -$$

$$-\sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} d_{\varkappa}^2(\alpha)\, \varepsilon^{\frac{2-\varkappa}{2}} \left(\sum_{\nu=2^{m-2}+1}^{2^{m-1}} \left(\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}}\right)^2 a_\nu^2\right)^{\varkappa/2} \geqq$$

$$\geqq d_1^{\varkappa}(\alpha) 2^{-(6+\varkappa+2\varkappa\alpha-\varkappa\gamma)} \sum_{m=1}^{\infty} \left\{\sum_{\nu=2^m+1}^{2^{m+1}} \nu^{2\gamma} a_\nu^2\right\}^{\varkappa/2},$$

i.e. the necessity of condition (1) is proved.

Next we prove that condition (1) is also sufficient. We suppose, as we may do without loss of generality, that $a_0 = a_1 = 0$. Applying (3), (6), and the Hölder inequality we have:

$$\sum_{n=3}^{\infty} n^{\varkappa\gamma+\varkappa-1} \int\limits_0^1 |\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x)|^{\varkappa}\,dx \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \left(\int\limits_0^1 |\sigma_{n+1}^{(\alpha)}(x) - \sigma_n^{(\alpha)}(x)|^2\,dx\right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+1}^{2^{m+1}} \left(\sum_{\nu=0}^{n} (L_{n,\nu}^{(\alpha)})^2 a_\nu^2 + \frac{1}{(A_{n+1}^{(\alpha)})^2} a_{n+1}^2\right)\right)^{\varkappa/2} =$$

$$= O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+1}^{2^{m+1}} \sum_{\nu=0}^{n} \frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}} \nu^2 a_\nu^2\right)^{\varkappa/2} +$$

$$+ O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+2}^{2^{m+1}+1} \frac{a_n^2}{(A_n^{(\alpha)})^2}\right)^{\varkappa/2} = O(1)\left(\sum\nolimits_1 + \sum\nolimits_2\right).$$

A standard computation shows that

$$\sum\nolimits_1 \leqq O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+1}^{2^{m+1}} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{\min(2^{l+1},n)} \frac{(n+1-\nu)^{2\alpha-2}\nu^2 a_\nu^2}{n^{2\alpha+2}} \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{m(2\gamma-1-2\alpha)} \sum_{n=2^m+1}^{2^{m+1}} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{\min(2^{l+1},n)} (n+1-\nu)^{2\alpha-2}\nu^2 a_\nu^2 \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{m(2\gamma-1-2\alpha)} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{2^{l+1}} \nu^2 a_\nu^2 \sum_{n=\max(2^m+1,\nu)}^{2^{m+1}} (n+1-\nu)^{2\alpha-2} \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{2m(\gamma-1)} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{2^{l+1}} \nu^2 a_\nu^2 \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{l=0}^{\infty} 2^{\varkappa l} \left(\sum_{\nu=2^l+1}^{2^{l+1}} a_\nu^2 \right)^{\varkappa/2} \sum_{m=l}^{\infty} 2^{m\varkappa(\gamma-1)} = O(1) \sum_{l=0}^{\infty} \left(\sum_{\nu=2^l+1}^{2^{l+1}} \nu^{2\gamma} a_\nu^2 \right)^{\varkappa/2}$$

and

$$\sum\nolimits_2 \leqq O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+1}^{2^{m+1}} \frac{a_n^2}{n^{2\alpha}} \right)^{\varkappa/2} + O(1) \sum_{m=0}^{\infty} 2^{\varkappa\gamma m+\varkappa m(\frac{1}{2}-\alpha)} a_{2^m+1}^{\varkappa} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{2\gamma m} \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right)^{\varkappa/2} \leqq O(1) \sum_{m=0}^{\infty} \left(\sum_{n=2^m+1}^{2^{m+1}} n^{2\gamma} a_n^2 \right)^{\varkappa/2}.$$

By the Beppo Levi theorem our proof is complete.

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On the divergence of rearranged Fourier series of square integrable functions

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Introduction

K. TANDORI [3] gave an elementary proof to the statement of A. N. KOLMOGOROFF [1] that there exists a square integrable function whose Fourier series can be rearranged so as to diverge almost everywhere. He [4] also proved the following theorem:

Theorem A. *If* $\{\varrho(n)\}$ *is a sequence of positive numbers with*

$$\varrho(n) = o(\sqrt{\log\log n}), \tag{1}$$

then there exists a sequence $\{c_n\}$ *with* $\Sigma c_n^2 \varrho^2(n) < \infty$ *such that the Walsh series* $\Sigma c_n w_n(x)$ *diverges almost everywhere in* $(0, 1)$ *in a certain rearrangement of its terms.*

Afterwards F. MÓRICZ [2] showed a generalization of [3] which can be considered as a trigonometric series analogue of Theorem A. That is:

Theorem B. *Suppose* (1). *Then there exists a square integrable function whose Fourier series* $\Sigma(a_n \cos nx + b_n \sin nx)$ *is such that*

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\varrho^2(n) < \infty \tag{2}$$

and which can be rearranged into an everywhere divergent series.

In the present paper we will sharpen Theorem B by refining the method of ist proof.

Theorem. *If* $\{\varrho(n)\}$ *is a sequence of positive numbers with*

$$\varrho(n) = o(\sqrt[4]{\log n}), \tag{3}$$

then there exists a square integrable function whose Fourier series fulfils (2) *and which can be rearranged into an everywhere divergent series.*

Corollary. *Suppose* (3), *then there exists a square integrable function whose Fourier series can be rearranged in such a way that the partial sums* $\sigma_N(x)$ *of the rearranged series satisfy*

$$\limsup_{N\to\infty} \frac{|\sigma_N(x)|}{\varrho(N)} > 0 \text{ everywhere.}$$

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§ 1. Lemmas

Consider a set $E = \bigcup_{i=1}^{m} J_i$ satisfying $\bar{J}_i \cap \bar{J}_j = \emptyset \ (i \neq j)$ [1]) and $\max_i |J_i| > 0$.[2]) If each J_i is an interval, then E is said to be *simple*, and we write $E \in \mathscr{S}$. More generally, if each J_i is either an interval or a point, then E is said to be *generalized simple*, and we write $E \in \mathscr{S}^*$. Suppose $E \in \mathscr{S}^*$, then for $0 < \varepsilon < \max_i |J_i|/2$, we set

$$E^{(\varepsilon)} = \bigcup_{\beta_i - \alpha_i > 2\varepsilon} [\alpha_i + \varepsilon, \beta_i - \varepsilon]$$

where α_i and β_i denote the left and right end points of J_i respectively. It is obvious that $E^{(\varepsilon)} \in \mathscr{S}$.

For a function $a_\nu \cos \nu x + b_\nu \sin \nu x \ (\not\equiv 0)$ we call ν its frequency. Two trigonometric polynomials are called disjoint if they have no terms of the same frequency.

$C_1, C_2, \ldots$ denote positive absolute constants which will be common in several lemmas.

Lemma 1. *Let* $E = \bigcup_i J_i \in \mathscr{S}^*$ *be a subset of* $[-\pi/12, \pi/12]$, $0 < \varepsilon < \max_i |J_i|/2$ and $0 < \eta \leqq 1$ *real numbers, and n a natural number such that* $n > C_1/\varepsilon\eta - 1$ $(C_1 = \pi)$. *Then there exists a non-negative trigonometric polynomial* $P(x)$ *with frequences* 6ν $(\nu = 0, 1, \ldots, n)$ *such that*

(7) $$P(x) \geqq 1 \quad \text{for} \quad x \in E^{(\varepsilon)},$$

(8) $$P(x) \leqq \eta \quad \text{for} \quad x \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - E$$

and

(9) $$\int_{-\pi}^{\pi} P^2(x)\,dx \leqq C_2 |E| \qquad \left(C_2 = \frac{27}{4}\pi^4\right).$$

[1]) $\bar{J}_i$ denotes the closure of J_i.

[2]) $|J_i|$ denotes the Lebesgue measure of J_i.

We can verify Lemma 1 with the aid of the proof of the similar lemma in [2], so we omit its proof.

Lemma 2. *Take the same assumptions and notations as in Lemma 1, an let $N\,(\geqq 12n+6)$ be a natural number divisible by 6. Furthermore set*

$$
\begin{aligned}
&Q_1(x) = (\cos Nx)\,P(x),\\
(10)\qquad &Q_2(x) = -C_3(\cos 3x)(\cos Nx)\,P(x) \qquad (C_3 = 2\sqrt{2}),\\
&Q_3(x) = -C_4(\cos 2Nx)\,P(x) \qquad (C_4 = 3+4\sqrt{2}).
\end{aligned}
$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequences 3ν having the following properties:

$$(11)\qquad N-6n-3 \leqq 3\nu \leqq N+6n+3 \quad or \quad 2N-6n \leqq 3\nu \leqq 2N+6n;$$

$$(12)\qquad |Q_1(x)+Q_2(x)+Q_3(x)| \leqq C_5\eta \quad for \quad x\in\left[-\frac{\pi}{12},\,\frac{\pi}{12}\right]-E$$

$$(C_5 = 1+C_3+C_4);$$

$$(13)\qquad \int_{-\pi}^{\pi} |Q_1(x)+Q_2(x)+Q_3(x)|^2\,dx \leqq C_6|E| \qquad (C_6 = C_2(1+C_3^2+C_4^2));$$

there exists a decomposition $E^{(\varepsilon)} = E_1+E_2+E_3$ such that

$$(14)\qquad \sum_{k=1}^{l} Q_k(x) \geqq \frac{1}{2} \quad for \quad x\in E_l \qquad (l=1,2,3).$$

In addition if

$$(15)\qquad \frac{2\pi}{N} \leqq \max_i |J_i| - 2\varepsilon$$

is satisfied, then each E_l contains an interval whose length is not smaller than $\pi/3N$ and $E_l \in \mathscr{S}^*$ $(l=1,2,3)$.

Proof. It is easy to see that the frequencies of the terms of $Q_1(x)$ and $Q_3(x)$ are divisible by 6, and those of $Q_2(x)$ divisible by 3 but not by 6. Moreover the frequencies 3ν of the terms of $Q_1(x)$ and $Q_2(x)$ satisfy the former inequalities of (11), and those of $Q_3(x)$ only the latter ones. (12) and (13) are shown by simple calculations using (8) and (9), respectively. And in virtue of (7), the following estimates hold:

$$Q_1(x) = (\cos Nx)\,P(x) \geqq \frac{1}{2}\cdot 1 = \frac{1}{2}$$

for

$$x \in E_1 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[\frac{1}{N}\left(2k\pi - \frac{\pi}{3}\right), \frac{1}{N}\left(2k\pi + \frac{\pi}{3}\right)\right];$$

$$Q_1(x) + Q_2(x) = (C_3 \cos 3x - 1)(-\cos Nx)P(x) \geqq \left(\frac{C_3}{\sqrt{2}} - 1\right) \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}$$

for

$$x \in E_2 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[\frac{1}{N}\left(2k\pi + \frac{2}{3}\pi\right), \frac{1}{N}\left(2k\pi + \frac{4}{3}\pi\right)\right];$$

$$Q_1(x) + Q_2(x) + Q_3(x) \geqq Q_3(x) - |Q_2(x)| - |Q_1(x)| > \frac{C_4}{2} - C_3 - 1 = \frac{1}{2}$$

for

$$x \in E_3 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left(\frac{1}{N}\left(k\pi + \frac{\pi}{3}\right), \frac{1}{N}\left(k\pi + \frac{2\pi}{3}\right)\right).$$

Now let us set $|J_{i_0}| = \max_i |J_i|$, and assume (15). Then in virtue of the definition of E_l $(l=1, 2, 3)$, each $E_l \cap J_{i_0}^{(\varepsilon)}$ contains an interval whose length is not smaller than $\pi/3N$. This completes the proof of Lemma 2.

Lemma 2′. *Let $P(x)$ be a trigonometric polynomial with frequencies 3ν $(\nu \leqq n)$, and $N (\geqq 6n+3)$ a natural number divisible by 3. Furthermore set*

$$(10') \qquad \begin{aligned} Q_1(x) &= (\cos Nx)P(x), \\ Q_2(x) &= -C_3(\cos x)(\cos Nx)P(x), \\ Q_3(x) &= -C_4(\cos 2Nx)P(x). \end{aligned}$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequencies ν having the following properties:

$$(11') \qquad N - 3n - 1 \leqq \nu \leqq 2N + 3n;$$

$$(13') \qquad \int_{-\pi}^{\pi} |Q_1(x) + Q_2(x) + Q_3(x)|^2 \, dx \leqq C_6 \int_{-\pi}^{\pi} P^2(x) \, dx;$$

for every set $E(\subset [-\pi/4, \pi/4])$ on which $P(x)$ is positive, there exists a decomposition $E = E_1 + E_2 + E_3$ such that

$$(14') \qquad \sum_{k=1}^{l} Q_k(x) \geqq \frac{P(x)}{2} \quad \text{for} \quad x \in E_l \qquad (l = 1, 2, 3)$$

and

$$(14'') \qquad Q_l(x) \geqq \frac{P(x)}{2} \quad \text{for} \quad x \in E_l \qquad (l = 1, 2, 3).$$

The proof of Lemma 2′ is quite in an analogy to that of Lemma 2.

Lemma 3. *If* $0 < \varepsilon < \pi/6$, *then there exist mutually disjoint trigonometric polynomials* $R_k^{(i)}(x)$ *and generalized simple sets*

$$E_k^{(i)} \subset [-\pi/12, \pi/12] \qquad (k=1, 2, \dots, 3^i; \quad i=0, 1, \dots)$$

with the following properties:

(i) *the frequencies occurring in* $R_k^{(i)}(x)$ $(k=1, 2, \dots, 3^i)$ *are divisible by* 3 *and smaller than* $6f_i(\varepsilon)$ *where*

$$f_i(\varepsilon) = \left(\frac{C_7}{\varepsilon}\right)^i 18^{\frac{i(i-1)}{2}} \qquad (C_7 = [128C_1C_5]+1);\ ^{3)}$$

(ii) $$\int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^i} R_k^{(i)}(x)\right)^2 dx \leqq C_8 \qquad \left(C_8 = C_6 \cdot \frac{\pi}{6}\right);$$

(iii) *the sets* $E_k^{(i)}$ $(k=1, 2, \dots, 3^i)$ *corresponding to the same value of* i *are disjoint;*

(iv) *set*

$$E_k^{(i)} = \bigcup_{j=1}^{g_k^{(i)}} J_j \quad \textit{and} \quad v_i(\varepsilon) = \sum_{k=1}^{3^i} g_k^{(i)},$$

then $v_i(\varepsilon) \leqq f_i(\varepsilon)$;

(v) *set*

$$F_i = \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - \bigcup_{k=1}^{3^i} E_k^{(i)},$$

then $|F_i| \leqq \varepsilon(1-1/2^i)$;

(vi) *the trigonometric polynomials* $R_k^{(j)}(x)$ $(k=1, \dots, 3^j;\ j=0, \dots, i)$ *can be arranged into a sequence*

$$(16) \qquad U_1^{(i)}(x), U_2^{(i)}(x), \dots, U_{h(i)}^{(i)}(x) \qquad (h(i) = (3^{i+1}-1)/2),$$

such that

$$(17) \qquad \sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) \geqq \frac{i+1}{4} \quad \textit{for} \quad x \in E_k^{(i)}$$

with $m_k^{(i)}$ *not depending on the particular point* $x \in E_k^{(i)}$ $(k=1, 2, \dots, 3^i)$.

Proof. Define $R_1^{(0)}(x)=1$ and $E_1^{(0)} = [-\pi/12, \pi/12]$, then these satisfy (i)—(vi) trivially. Setting $\varkappa_i = 1/2^{i+2}f_i(\varepsilon)$ $(i \geqq 0)$, $\eta_0 = 1$ and $\eta_i = 1/C_5 4(3^i-1)$ $(i \geqq 1)$, we take natural numbers

$$n_i = \left[\frac{C_1}{\varkappa_i \varepsilon \eta_i}\right] \qquad (i=0, 1, \dots)$$

and

$$N_k^{(i)} = (2k-1)(12n_i+6) \qquad (k=1, 2, \dots, 3^i; \quad i=0, 1, \dots).$$

[3]) The integer part of a real number α is denoted by $[\alpha]$.

We have the following estimates:

$$(18)\quad N_1^{(i)}-6n_i-3=6n_i+3>6\left(\frac{C_1}{\varkappa_i\cdot\frac{\pi}{6}\cdot 1}-1\right)+3=36\cdot 2^{i+2}f_i(\varepsilon)-3>72f_i(\varepsilon)\qquad (i\geqq 0);$$

$$(19)\quad 2N_{3^i}^{(i)}+6n_i=\{24(2\cdot 3^i-1)+6\}n_i+12(2\cdot 3^i-1)\leqq$$

$$\leqq(48\cdot 3^i-18)\frac{C_1}{\varepsilon}2^{i+2}f_i(\varepsilon)C_5 4\cdot 3^i+24\cdot 3^i-12=$$

$$=6\cdot 18^i\frac{128C_1C_5}{\varepsilon}f_i(\varepsilon)-6^i\frac{288C_1C_5}{\varepsilon}f_i(\varepsilon)+24\cdot 3^i-12<$$

$$<6\cdot 18^i\left(\frac{C_7}{\varepsilon}\right)f_i(\varepsilon)=6\cdot f_{i+1}(\varepsilon)\qquad (i\geqq 0);$$

$$(20)\qquad 2\varkappa_i\varepsilon=\frac{2\varepsilon}{2^{i+2}f_i(\varepsilon)}<\begin{cases}\dfrac{\pi}{12} & (i=0),\\[2ex] \dfrac{\pi}{24f_i(\varepsilon)}<\dfrac{\pi}{8N_{3^{i-1}}^{(i-1)}} & (i\geqq 1).\end{cases}$$

Applying Lemma 2 to ($E_1^{(0)}$, $\varkappa_0\varepsilon$, η_0, n_0 and $N_1^{(0)}$), we get the mutually disjoint trigonometric polynomials $Q_l(x)$ $(l=1, 2, 3)$ and the decomposition $E_1^{(0)(\varkappa_0\varepsilon)}=E_1+E_2+E_3$. Define $R_k^{(1)}(x)=Q_k(x)$ and $E_k^{(1)}=E_k$ $(k=1, 2, 3)$. Then we can easily check that (i)—(vi) hold for $i=1$. For example as to (iv),

$$v_1(\varepsilon)\leqq v_0(\varepsilon)+2\left(\left[\frac{\frac{\pi}{12}-\frac{\varepsilon}{4}}{\frac{\pi}{2N_1^{(0)}}}\right]+1\right)\leqq$$

$$\leqq 3+N_1^{(0)}\left(\frac{1}{3}-\frac{\varepsilon}{\pi}\right)<3+3f_1(\varepsilon)\left(\frac{1}{3}-\frac{\varepsilon}{\pi}\right)\leqq f_1(\varepsilon)+3-3\cdot 128C_5<f_1(\varepsilon);$$

and as to (vi), we set $U_1^{(1)}(x)=R_1^{(0)}(x)$, $U_2^{(1)}(x)=R_1^{(1)}(x)$, $U_3^{(1)}(x)=R_2^{(1)}(x)$, $U_4^{(1)}(x)=R_3^{(1)}(x)$ and $m_k^{(1)}=1+k$ $(k=1, 2, 3)$. Furthermore, since $|E_1^{(0)}|-2\varkappa_0\varepsilon>\pi/6-\pi/12>2\pi/N_1^{(0)}$, we see that each $E_k^{(1)}$ $(k=1, 2, 3)$ contains an interval whose length is not smaller than $\pi/2N_1^{(0)}$ and that $E_k^{(1)}\in\mathscr{S}^*$.

Now we suppose that $R_k^{(j)}(x)$ and $E_k^{(j)}$ $(k=1, \dots, 3^j;\ j=0, \dots, i)$ are already defined and satisfy (i)—(vi), and that

$$(21)\qquad \max_{1\leqq j\leqq g_k^{(i)}}|J_j|\geqq\frac{\pi}{3N_{3^{i-1}}^{(i-1)}}$$

for each $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$. Then by (20) and (21).

$$2\varkappa_i\varepsilon < \max_{1\leq j\leq g_k^{(i)}} |J_j|$$

holds for each $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$. By the application of Lemma 2 to each $(E_k^{(i)}, \varkappa_i\varepsilon, \eta_i, n_i$ and $N_k^{(i)})$ $(k=1, 2, \ldots, 3^i)$, we define the mutually disjoint trigonometric polynomials

$$\begin{aligned} R_{3k-2}^{(i+1)}(x) &= (\cos N_k^{(i)}x)P_k^{(i)}(x), \\ R_{3k-1}^{(i+1)}(x) &= -C_3(\cos 3x)(\cos N_k^{(i)}x)P_k^{(i)}(x), \\ R_{3k}^{(i+1)}(x) &= -C_4(\cos 2N_k^{(i)}x)P_k^{(i)}(x) \end{aligned} \tag{22}$$

and the decompositions

$$E_k^{(i)(\varkappa_i\varepsilon)} = E_{3k-2}^{(i+1)} + E_{3k-1}^{(i+1)} + E_{3k}^{(i+1)} \qquad (k=1, 2, \ldots, 3^i).$$

In virtue of (11), the frequencies of (22) belong to $A_k \cup B_k$ where

$$\begin{aligned} A_k &= [(2k+1)N_1^{(i)} - 6n_i - 3, (2k+1)N_1^{(i)} + 6n_i + 3], \\ B_k &= [2(2k+1)N_1^{(i)} - 6n_i, 2(2k+1)N_1^{(i)} + 6n_i]. \end{aligned}$$

It is obvious that $A_k \cap A_{k'} = \emptyset$ and $B_k \cap B_{k'} = \emptyset$ for $k \neq k'$. Moreover we have $A_k \cap B_{k'} = \emptyset$ $(k \neq k')$ since, though $|A_k|/2 + |B|_{k'}/2 = (6n_i+3) + 6n_i < N_1^{(i)}$ holds, the distance of the middle points of A_k and $B_{k'}$ is not smaller than $N_1^{(i)}$. Thus the trigonometric polynomials $R_k^{(i+1)}(x)$ $(k=1, 2, \ldots, 3^{i+1})$ are mutually disjoint. And we are going to show (i)—(vi) and (21) replacing i with $i+1$.

By (18) the frequencies occurring in $R_1^{(i+1)}(x)$ are larger than those of $R_k^{(i)}(x)$ $(k=1, 2, \ldots, 3^i)$, and by (19) the property (i) is verified. As to (ii), we have

$$\int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^{i+1}} R_k^{(i+1)}(x)\right)^2 dx = \sum_{k=1}^{3^i} \int_{-\pi}^{\pi} (R_{3k-2}^{(i+1)}(x) + R_{3k-1}^{(i+1)}(x) + R_{3k}^{(i+1)}(x))^2\, dx =$$

$$= \sum_{k=1}^{3^i} C_6 |E_k^{(i)}| \leq C_6 \cdot \frac{\pi}{6} = C_8;$$

and as to (iii), it is obvious. As to (iv), setting

$$E_k^{(i)(\varkappa_i\varepsilon)} = \bigcup_l I_l^{(i,k)} \qquad (k=1, 2, \ldots, 3^i),$$

we have
$$v_{i+1}(\varepsilon) \leq v_i(\varepsilon) + \sum_{k=1}^{3^i} \sum_l 2\left(\left[\frac{|I_l^{(i,k)}|}{\frac{\pi}{N_k^{(i)}}}\right] + 1\right)$$

$$\leq v_i(\varepsilon) + \frac{2N_{3^i}^{(i)}}{\pi} \sum_{k=1}^{3^i} |E_k^{(i)(\varkappa_i\varepsilon)}| + 2v_i(\varepsilon) \leq 3f_i(\varepsilon) + \frac{6f_{i+1}(\varepsilon)}{\pi}\left(\frac{\pi}{6} - \frac{\varepsilon}{2}\right) < f_{i+1}(\varepsilon);$$

and as to (v),

$$|F_{i+1}| = \left|F_i \cup \left\{\bigcup_{k=1}^{3^i} (E_k^{(i)} - E_k^{(i)(\varkappa_i \varepsilon)})\right\}\right| \leqq |F_i| + 2\varkappa_i \varepsilon \cdot v_i(\varepsilon) \leqq$$

$$\leqq \varepsilon\left(1 - \frac{1}{2^i}\right) + \frac{2\varepsilon}{2^{i+2} f_i(\varepsilon)} f_i(\varepsilon) = \varepsilon\left(1 - \frac{1}{2^{i+1}}\right).$$

As to (vi), we define the sequence

$$U_1^{(i+1)}(x), U_2^{(i+1)}(x), \ldots, U_{h(i+1)}^{(i+1)}(x)$$

by inserting $R_{3k-2}^{(i+1)}(x)$, $R_{3k-1}^{(i+1)}(x)$, $R_{3k}^{(i+1)}(x)$ after $R_k^{(i)}(x)$ $(k = 1, 2, \ldots, 3^i)$ in (16), and define $m_k^{(i+1)}$ $(k = 1, 2, \ldots, 3^{i+1})$ by

$$U_{m_k^{(i+1)}}^{(i+1)}(x) = R_k^{(i+1)}(x) \qquad (k = 1, 2, \ldots, 3^{i+1}).$$

Then if $x \in E_{3k-3+l}^{(i+1)}$, $1 \leqq k \leqq 3^i$ and $1 \leqq l \leqq 3$, we obtain

$$\sum_{j=1}^{m_{3k-3+l}^{(i+1)}} U_j^{(i+1)}(x) = \sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) + \sum_{j=1}^{3k-3+l} R_j^{(i+1)}(x) \geqq$$

$$\geqq \sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) + \sum_{j=1}^{l} R_{3k-3+j}^{(i+1)}(x) - \sum_{j=1}^{3^i}{}' \, |R_{3j-2}^{(i+1)}(x) + R_{3j-1}^{(i+1)}(x) + R_{3j}^{(i+1)}(x)| \geqq$$

$$\geqq \frac{i+1}{4} + \frac{1}{2} - (3^i - 1) C_5 \eta_i = \frac{(i+1)+1}{4}.$$

Finally by

$$\frac{2\pi}{N_1^{(i)}} < \frac{2\pi}{72 f_i(\varepsilon)} < \frac{\frac{\pi}{3} - \frac{\pi}{8}}{N_{3^i-1}^{(i-1)}} < \max_{1 \leqq j \leqq g_k^{(i)}} |J_j| - 2\varkappa_i \varepsilon,$$

we get (21) for each $E_k^{(i+1)}$ $(k = 1, 2, \ldots, i+1)$. Thus the statement of Lemma 3 is proved.

Lemma 4. *There exist mutually disjoint trigonometric polynomials* $S_j^{(i)}(x)$ $(j = 1, 2, \ldots, 3h(i)+3;\ i = C_9, C_9+1, \ldots)$[4]) *with the following properties:*

(vii) *the frequencies* ν *occurring in* $S_j^{(i)}(x)$ *satisfy* $5^{i^2} \leqq \nu \leqq 5^{i^2+1}$;

$$\text{(viii)} \qquad \int_{-\pi}^{\pi} \left(\sum_{j=1}^{3h(i)+3} S_j^{(i)}(x)\right)^2 dx \leqq \frac{C_{10}}{i+1} \qquad \left(C_{10} = C_6\left(C_8 + \frac{C_2}{8}\right)\right);$$

$$\text{(ix)} \qquad \sum_{j=\mu_1^{(i)}(x)}^{\mu_2^{(i)}(x)} S_j^{(i)}(x) \geqq \frac{1}{8} \quad \textit{for} \quad 0 \leqq x \leqq \frac{\pi}{12},$$

where $1 \leqq \mu_1^{(i)}(x) \leqq \mu_2^{(i)}(x) \leqq 3h(i)+3$.

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[4]) C_9 will be defined later on, see (26).

Proof. Fix the natural number i, and apply Lemma 3 to $\varepsilon_i = 1/(i+1)$. Then we get the mutually disjoint trigonometric polynomials $U_j^{(i)}(x)$ $(j=1, 2, \ldots, h(i))$ and the simple sets $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$. It is obvious that the frequencies occurring in $U_j^{(i)}(x)$ are smaller than $6f_i(\varepsilon_i)$, and that (17) and

$$\sum_{j=1}^{h(i)} \int_{-\pi}^{\pi} (U_j^{(i)}(x))^2\,dx \leqq C_8(i+1) \tag{23}$$

hold. In view of (iv), $E_k^{(i-1)}$ consists of $g_k^{(i-1)}$ disjoint intervals, therefore $E_k^{(i-1)(\varkappa_{i-1}\varepsilon_i)}$ consists of at most $g_k^{(i-1)}$ disjoint intervals too. Hence

$$F_i = \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - \bigcup_{k=1}^{3^i} E_k^{(i)} = \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - \bigcup_{k=1}^{3^{i-1}} E_k^{(i-1)(\varkappa_{i-1}\varepsilon_i)}$$

consists of at most $v_{i-1}(\varepsilon_i)+1$ disjoint intervals.

Let $H_i \subset [0, \pi/6]$ be the symmetric set defined by $H_i \cap [0, \pi/12] = F_i \cap [0, \pi/12]$, then H_i consists of at most $f_{i-1}(\varepsilon_i)$ disjoint intervals. Setting $H_i = \Sigma[\alpha, \beta]$, $\varepsilon_i' = \varepsilon_i/2f_{i-1}(\varepsilon_i)$ and $H_i' = [\alpha-\varepsilon_i', \beta+\varepsilon_i']$, we see that $H_i' \subset [0, \pi/6]$, $H_i' \in \mathcal{S}$ and

$$|H_i'| \leqq |H_i| + 2\varepsilon_i' f_{i-1}(\varepsilon_i) \leqq \varepsilon_i\left(1-\frac{1}{2^i}\right) + \varepsilon_i \leqq \frac{2}{i+1}.$$

Applying Lemma 1 to (H_i', ε_i', 1 and $[C_1/\varepsilon_i']$), we get the trigonometric polynomial $P^{(i)}(x)$ with frequencies 6ν $(\nu = 0, 1, \ldots, [C_1/\varepsilon_i'])$ such that

$$P^{(i)}(x) \geqq 1 \quad \text{for} \quad x \in H_i \subset H_i'^{(\varepsilon_i')} \tag{24}$$

and

$$\int_{-\pi}^{\pi} (P^{(i)}(x))^2\,dx \leqq C_2|H_i'| \leqq \frac{2C_2}{i+1}. \tag{25}$$

Now we suppose $i \geqq C_9$ so that the inequality

$$37f_i(\varepsilon_i) = 37C_7^i(i+1)^i 18^{\frac{i(i-1)}{2}} = \tag{26}$$

$$= 18^{\left(\frac{1}{2}+\lambda\right)i^2 - \lambda i^2 + i\left\{\log_{18}(i+1) + \log_{18} C_7 - \frac{1}{2}\right\} + \log_{18} 37} \leqq 18^{\left(\frac{1}{2}+\lambda\right)i^2} = 5^{i^2}.$$

may hold. Setting $N_1 = 5^{i^2} + 6f_i(\varepsilon_i) + (3+(-1)^i)/2$ and $N_2 = 2N_1 + 6f_i(\varepsilon_i) + 6[C_1/\varepsilon_i'] + 3$, we define

$$S_j^{(i)}(x) = (\cos N_1 x)\frac{U_j^{(i)}(x)}{i+1},$$

$$S_{h(i)+j}^{(i)}(x) = -C_3(\cos x)(\cos N_1 x)\frac{U_j^{(i)}(x)}{i+1},$$

$$S_{2h(i)+j}^{(i)}(x) = -C_4(\cos 2N_1 x)\frac{U_j^{(i)}(x)}{i+1}$$

$(j=1, 2, \ldots, h(i))$, and

$$S^{(i)}_{3h(i)+1}(x) = (\cos N_2 x)\frac{P^{(i)}(x)}{4},$$

$$S^{(i)}_{3h(i)+2}(x) = -C_3(\cos x)(\cos N_2 x)\frac{P^{(i)}(x)}{4},$$

$$S^{(i)}_{3h(i)+3}(x) = -C_4(\cos 2N_2 x)\frac{P^{(i)}(x)}{4}.$$

Then using (11′) we easily see that $S_j^{(i)}(x)$ $\left(j = 1, 2, \ldots, 3h(i)+3\right)$ are mutually disjoint trigonometric polynomials with frequencies ν satisfying

$$5^{i^2} \leqq N_1 - 6f_i(\varepsilon_i) - 1 \leqq \nu \leqq 2N_2 + 6\left[\frac{C_1}{\varepsilon_i'}\right].$$

And by (26),

$$2N_2 + 6\left[\frac{C_1}{\varepsilon_i'}\right] = 4\cdot 5^{i^2} + 36f_i(\varepsilon_i) + 18\left[\frac{C_1}{\varepsilon_i'}\right] + 18 \leqq 4\cdot 5^{i^2} + 37f_i(\varepsilon_i) \leqq 5^{i^2+1}.$$

By (13′), (23) and (25), we obtain

$$\int_{-\pi}^{\pi}\left(\sum_{j=1}^{3h(i)+3} S_j^{(i)}(x)\right)^2 dx =$$

$$= \sum_{j=1}^{h(i)} \int_{-\pi}^{\pi} |S_j^{(i)}(x) + S^{(i)}_{h(i)+j}(x) + S^{(i)}_{2h(i)+j}(x)|^2\, dx + \int_{-\pi}^{\pi}\left|\sum_{l=1}^{3} S^{(i)}_{3h(i)+l}(x)\right|^2 dx \leqq$$

$$\leqq \sum_{j=1}^{h(i)} C_6 \int_{-\pi}^{\pi}\left(\frac{U_j^{(i)}(x)}{i+1}\right)^2 dx + C_6\int_{-\pi}^{\pi}\left(\frac{P^{(i)}(x)}{4}\right)^2 dx \leqq$$

$$\leqq \frac{C_6}{(i+1)^2}\cdot C_8(i+1) + \frac{C_6}{16}\cdot\frac{2C_2}{i+1} = \frac{C_{10}}{i+1}.$$

To prove (ix), suppose $0 \leqq x \leqq \pi/12$. Then $x \in \bigcup_{k=1}^{3^i} E_k^{(i)}$ or $x \in H_i$. We set $\mu_1^{(i)}(x) = 1$ and $\mu_2^{(i)}(x) = m_k^{(i)}$ for

$$x \in E_k^{(i)} \cap \bigcup_{j=-\infty}^{\infty}\left[\frac{1}{N_1}\left(2j\pi - \frac{\pi}{3}\right), \frac{1}{N_1}\left(2j\pi + \frac{\pi}{3}\right)\right];$$

$\mu_1^{(i)}(x) = h(i)+1$ and $\mu_2^{(i)}(x) = h(i) + m_k^{(i)}$ for

$$x \in E_k^{(i)} \cap \bigcup_{j=-\infty}^{\infty}\left[\frac{1}{N_1}\left(2j\pi + \frac{2}{3}\right), \frac{1}{N_1}\left(2j\pi + \frac{4}{3}\pi\right)\right];$$

and $\mu_1^{(i)}(x) = 2h(i)+1$ and $\mu_2^{(i)}(x) = 2h(i)+m_k^{(i)}$ for

$$x \in E_k^{(i)} \cap \bigcup_{j=-\infty}^{\infty} \left(\frac{1}{N_1}\left(j\pi+\frac{\pi}{3}\right), \frac{1}{N_1}\left(j\pi+\frac{2}{3}\pi\right)\right).$$

Hence in the case of $x \in E_k^{(i)}$, we get

$$\sum_{j=\mu_1^{(i)}(x)}^{\mu_2^{(i)}(x)} S_j^{(i)}(x) = \begin{cases} \dfrac{\cos N_1 x}{i+1} \displaystyle\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) \\ \text{or } \dfrac{-C_3(\cos x)(\cos N_1 x)}{i+1} \displaystyle\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) \\ \text{or } \dfrac{-C_4(\cos 2N_1 x)}{i+1} \displaystyle\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x). \end{cases}$$

Now using (14″) and (17),

$$\sum_{j=\mu_1^{(i)}(x)}^{\mu_2^{(i)}(x)} S_j^{(i)}(x) \geqq \frac{1}{i+1} \cdot \frac{\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x)}{2} \geqq \frac{1}{8}.$$

In the case of $x \in H_i$, we set $\mu_1^{(i)}(x) = 3h(i)+1$ and $\mu_2^{(i)}(x) = 3h(i)+1$ or $3h(i)+2$ or $3h(i)+3$. Then using (14′) and (24), we get the assertion of (ix). So the proof of Lemma 4 is complete.

§ 2. Proof of the theorem

Define the sequence of natural numbers $(C_9 \leqq) m_1 < m_2 < \cdots$ such that

(27) $$\frac{\varrho(n)}{\sqrt[4]{\log_5 n}} \leqq \frac{1}{k} \quad \text{if} \quad n \geqq 5^{m_k^2}.$$

Then by (vii), setting

(28) $$T_k(x) = \sum_{j=1}^{3h(m_k)+3} S_j^{(m_k)}\left(x - \frac{(k)_{24}\pi}{12}\right) =$$ [5]

$$= \sum_{n=5^{m_k^2}}^{5^{m_k^2}+1} (a_n \cos nx + b_n \sin nx) = \sum_{n=5^{m_k^2}}^{5^{m_{k+1}^2}-1} (a_n \cos nx + b_n \sin nx) \qquad (k = 1, 2, \ldots),$$

[5]) $(k)_{24}$ denotes the remainder of k modulo 24.

we consider the series $\sum_1^\infty (a_n \cos nx + b_n \sin nx)$. And we define the rearrangement $\{n_j\}$ by

$$S_1^{(m_1)}\left(x-\frac{\pi}{12}\right)+S_2^{(m_1)}\left(x-\frac{\pi}{12}\right)+\cdots+S_{3h(m_1)+3}^{(m_1)}\left(x-\frac{\pi}{12}\right)+$$

$$+S_1^{(m_2)}\left(x-\frac{2\pi}{12}\right)+\cdots+S_{3h(m_2)+3}^{(m_2)}\left(x-\frac{2\pi}{12}\right)+\cdots+S_j^{(m_k)}\left(x-\frac{(k)_{24}\pi}{12}\right)+\cdots$$

which diverges everywhere in virtue of (ix). By (27), (28) and (viii), we get

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)\varrho^2(n) \leqq \sum_{k=1}^{\infty}\frac{\sqrt{m_k^2+1}}{k^2}\sum_{n=5^{m_k^2}}^{5^{m_k^2+1}}(a_n^2+b_n^2) =$$

$$=\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sqrt{m_k^2+1}}{k^2}\int_{-\pi}^{\pi}T_k^2(x)\,dx \leqq \frac{C_{10}}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.$$

Thus, in accordance with the Riesz—Fischer theorem, the assertion of our theorem is proved.

Next define $(A_n \cos nx + B_n \sin nx)$ by

$$A_n=\frac{a_n\sqrt{m_k+1}}{k},\quad B_n=\frac{b_n\sqrt{m_k+1}}{k}\qquad (5^{m_k^2}\leqq n<5^{m_{k+1}^2};\ k\geqq 1),$$

and the proof of Corollary runs similarly to that of Theorem 2 in [2].

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Über eine neue Klasse von Mittelwerten

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§ 1. Einleitung

Die bekannte Minkowskische Ungleichung behauptet, daß

$$(1)\qquad \left(\frac{\sum_{i=1}^{n}(x_i+y_i)^p}{n}\right)^{1/p} \leqq \left(\frac{\sum_{i=1}^{n}x_i^p}{n}\right)^{1/p} + \left(\frac{\sum_{i=1}^{n}y_i^p}{n}\right)^{1/p} \qquad \text{für} \quad x_i, y_i \in [0, \infty);\ p \geqq 1.$$

Eine Verallgemeinerung von (1) ist die Ungleichung

$$(2)\qquad M_\varphi\big(k(x_i, y_i)\big)_f \leqq k\big(M_\psi(x_i)_g, M_\chi(y_i)_h\big),$$

wobei φ, ψ, χ streng monotone stetige Funktionen, f, g, h positive Funktionen auf einem Intervall I sind, $k: I \times I \to I$ und

$$(3)\qquad M_\varphi(x_1, \ldots, x_n)_f = M_\varphi(x_i)_f = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} f(x_i)\varphi(x_i)}{\sum_{i=1}^{n} f(x_i)}\right).$$

In [11] haben wir (unter geeigneten Bedingungen) notwendige und hinreichende Bedingungen für (2) gegeben. Ebenda haben wir spezielle Ungleichungen vom Typ (2) untersucht (s. noch [4], [5], [8]).

In dieser Arbeit werden die allgemeineren Mittelwerte

$$(4)\qquad M_\varphi(x_1, \ldots, x_n)_{[f_1, \ldots, f_n]} = M_\varphi(x_i)_{f_i} = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} f_i(x_i)\varphi(x_i)}{\sum_{i=1}^{n} f_i(x_i)}\right)$$

studiert, wobei $f_1, f_2, \ldots$ positive Funktionen und φ eine streng monotone stetige Funktion sind. In § 2 geben wir (unter gewissen Voraussetzungen) notwendige und hinreichende Bedingungen für die Ungleichung

$$M_\varphi\big(k(x_i, y_i)\big)_{f_i} \leqq k\big(M_\psi(x_i)_{g_i}, M_\chi(y_i)_{h_i}\big).$$

In § 3 lösen wir das Gleichheitsproblem der Mittelwerte (4), während wir in § 4 alle quasihomogenen Mittelwerte vom Typ (4) bestimmen.

§ 2. Eine allgemeine Ungleichung

Es sei I ein beliebiges Intervall, R bzw. R_+ die Mengen der reellen bzw. positiven reellen Zahlen. Der Kürze halber führen wir die folgenden Bezeichnungen ein:

$Q(I)=\{f|f:I\to R_+\}$,

$D(I)=\{\varphi|\varphi:I\to R,\ \varphi$ differenzierbar auf $I,\ \varphi'(x)\neq 0$ für $x\in I\}$,

$D(I^2)=\{k|k:I\times I\to I,\ k$ total differenzierbar auf $I\times I\}$.

Liegt φ in $D(I)$, so ist φ streng monoton.

Satz 1. *Es seien* $\varphi, \psi, \chi\in D(I)$, $f_j, g_j, h_j\in Q(I)$ $(j=1, 2, \ldots)$, $k\in D(I^2)$. *Wir setzen ferner voraus, daß*

(A) *die Reihen* $\sum_{j=1}^{\infty} g_j(x)$, $\sum_{j=1}^{\infty} h_j(x)$ *für* $x\in I$ *divergent sind,*

(B) *die (endlichen) Grenzfunktionen*

$$\lim_{m\to\infty}\frac{\sum_{j=1}^{m} f_j(k(t,s))}{\sum_{j=1}^{m} g_j(t)} = G(t,s), \quad \lim_{m\to\infty}\frac{\sum_{j=1}^{m} f_j(k(t,s))}{\sum_{j=1}^{m} h_j(s)} = H(t,s)$$

für $t, s\in I$ *existieren. Die Ungleichung*

$$M_\varphi(k(x_i, y_i))_{f_i} \leqq k(M_\psi(x_i)_{g_i}, M_\chi(y_i)_{h_i}) \tag{5}$$

$(x_i, y_i\in I;\ n=2, 3, \ldots)$ *gilt dann und nur dann, wenn die Ungleichungen*

$$\frac{\varphi(k(u,v))-\varphi(k(t,s))}{\varphi'(k(t,s))} f_i(k(u,v)) \leqq \tag{6}$$

$$\leqq \frac{\psi(u)-\psi(t)}{\psi'(t)} g_i(u) G(t,s) k_t(t,s) + \frac{\chi(v)-\chi(s)}{\chi'(s)} h_i(v) H(t,s) k_s(t,s)$$

für alle $u, v, t, s\in I$ $(i=1, 2, \ldots)$ *erfüllt sind.*

Beweis. *Notwendigkeit.* Substituieren wir in (5) $n\geqq i$; $x_i=u$, $x_1=\cdots=x_{i-1}=x_{i+1}=\cdots=x_n=t$; $y_i=v$, $y_1=\cdots=y_{i-1}=y_{i+1}=\cdots=y_n=s$. Ist φ eine wachsende Funktion, so erhalten wir

$$f_i(k(u,v))[\varphi(k(u,v))-\varphi(k(T_n, S_n))] \leqq \tag{7}$$

$$\leqq {\sum_{j=1}^{n}}' f_j(k(t,s))[\varphi(k(T_n, S_n))-\varphi(k(t,s))],$$

wo

$$T_n = \psi^{-1}\left(\frac{g_i(u)\psi(u)+\sum_{j=1}^{n}{}' g_j(t)\psi(t)}{g_i(u)+\sum_{j=1}^{n}{}' g_j(t)}\right), \tag{8}$$

$$S_n = \chi^{-1}\left(\frac{h_i(v)\chi(v)+\sum_{j=1}^{n}{}' h_j(s)\chi(s)}{h_i(v)+\sum_{j=1}^{n}{}' h_j(s)}\right)$$

und $\sum_{j=1}^{n}{}' a_j = \sum_{j=1}^{n} a_j - a_i$. Da $\varphi(x)$ und $k(t, s)$ differenzierbare Funktionen sind, gilt:

$$\varphi(k(T_n, S_n)) - \varphi(k(t, s)) = \tag{9}$$

$$=[\varphi'(k(t, s))+\omega_1][(k_t(t, s)+\omega_2)(T_n-t)+(k_s(t, s)+\omega_3)(S_n-s)],$$

wo $\omega_i \to 0$ $(i=1, 2, 3)$ für $(T_n, S_n)\to(t, s)$. Wegen der Bedingung (A) hat man $\lim_{n\to\infty} T_n=t$, $\lim_{n\to\infty} S_n=s$, die Funktionen ω_i genügen also der Relation $\lim_{n\to\infty} \omega_i=0$ $(i=1, 2, 3)$.

Es sei erstens $u\neq t$, dann ist $T_n\neq t$ und

$$\lim_{n\to\infty} (T_n-t) \sum_{j=1}^{n}{}' f_j(k(t, s)) = \tag{10}$$

$$= \lim_{n\to\infty} \frac{g_i(u) \sum_{j=1}^{n}{}' f_j(k(t, s))}{g_i(u)+\sum_{j=1}^{n}{}' g_j(t)} \cdot \frac{\psi(u)-\psi(t)}{\frac{\psi(T_n)-\psi(t)}{T_n-t}} = \frac{\psi(u)-\psi(t)}{\psi'(t)} G(t, s)g_i(u).$$

Für $u=t$ ist diese Limesrelation offenbar richtig. Genauso erhalten wir, daß

$$\lim_{n\to\infty} (S_n-s) \sum_{j=1}^{n}{}' f_j(k(t, s)) = \frac{\chi(v)-\chi(s)}{\chi'(s)} H(t, s)h_i(v) \tag{11}$$

gilt. Auf Grund von (9), (10), (11) folgt (6) aus (7) mit $n\to\infty$. Im Falle einer abnehmenden Funktion φ verläuft der Beweis ebenso.

Hinlänglichkeit. Setzen wir in (6) nacheinander

$$u=x_i, \quad v=y_i, \quad t=M_\psi(x_i)_{g_i}, \quad s=M_\chi(y_i)_{h_i} \qquad (i=1, \dots, n)$$

und addieren die gewonnenen Ungleichungen, so erhalten wir

$$\frac{\varphi(M_\varphi(k(x_i, y_i))_{f_i})-\varphi(k(t, s))}{\varphi'(k(t, s))} \leqq 0, \tag{12}$$

woraus (5) folgt.

Bemerkungen. 1. Steht in (5), (6) $\geqq$ oder $=$ statt $\leqq$, so ist der Satz 1 auch gültig.

2. Es bezeichne E_i die Menge derjenigen (u, v, t, s), für welche in (6) das Gleichheitszeichen besteht. Aus dem Beweis der Hinlänglichkeit kann man einsehen, daß die Gleichheit in (5) genau dann besteht, wenn $(x_i, y_i, t, s)\in E_i$ für $i=1, 2, \dots, n$ gilt, wobei $t=M_\psi(x_i)_{g_i}$, $s=M_\chi(y_i)_{h_i}$.

3. Mit $k(x, y)\equiv x$ erhalten wir aus dem Satz 1 notwendige und hinreichende Bedingungen für die Ungleichung

$$M_\varphi(x_i)_{f_i} \leqq M_\psi(x_i)_{g_i} \qquad (x_i\in I;\ n=2, 3, \dots). \tag{13}$$

Im Spezialfall $f_i=f$, $g_i=g$ $(i=1, 2, \dots)$ wurde diese Ungleichung in [6], [7], [10] untersucht.

In Satz 1 kann die Gestalt der Funktionen $G(t, s)$, $H(t, s)$ sehr kompliziert sein. Hier geben wir einen Spezialfall an, bei welchem diese Funktionen einfach ausgerechnet werden können.

Satz 2. *Es seien $\varphi, \psi, \chi\in D(I)$, $f_j, g_j, h_j\in Q(I)$ $(j=1, 2, \dots)$, $k\in D(I^2)$. Wir setzen ferner voraus, daß die (endlichen) Limesfunktionen*

$$\lim_{m\to\infty} f_m(x) = f(x), \quad \lim_{m\to\infty} g_m(x) = g(x), \quad \lim_{m\to\infty} h_m(x) = h(x)$$

existieren, und in $Q(I)$ liegen. Die Ungleichung

$$M_\varphi\big(k(x_i, y_i)\big)_{f_i} \leqq k\big(M_\psi(x_i)_{g_i}, M_\chi(y_i)_{h_i}\big)$$

$(x_i, y_i\in I;\ n=2, 3, \dots)$ gilt genau dann, wenn die Ungleichungen

$$\frac{\varphi\big(k(u, v)\big)-\varphi\big(k(t, s)\big)}{\varphi'\big(k(t, s)\big)}\,\frac{f_i\big(k(u, v)\big)}{f\big(k(t, s)\big)} \leqq$$

$$\leqq \frac{\psi(u)-\psi(t)}{\psi'(t)}\,\frac{g_i(u)}{g(t)}\,k_t(t, s)+\frac{\chi(v)-\chi(s)}{\chi'(s)}\,\frac{h_i(v)}{h(s)}\,k_s(t, s)$$

$(u, v, t, s\in I;\ i=1, 2, \dots)$ erfüllt sind.

Zum Beweis nehmen wir in Betracht, daß nach dem Satz von O. Stolz (siehe z. B. [9]) die Relationen

$$G(t, s) = \lim_{m\to\infty} \frac{\sum\limits_{j=1}^{m} f_j\big(k(t, s)\big)}{\sum\limits_{j=1}^{m} g_j(t)} = \lim_{m\to\infty} \frac{f_m\big(k(t, s)\big)}{g_m(t)} = \frac{f\big(k(t, s)\big)}{g(t)},$$

$$H(t, s) = \lim_{m\to\infty} \frac{\sum\limits_{j=1}^{m} f_j\big(k(t, s)\big)}{\sum\limits_{j=1}^{m} h_j(s)} = \lim_{m\to\infty} \frac{f_m\big(k(t, s)\big)}{h_m(s)} = \frac{f\big(k(t, s)\big)}{h(s)}$$

gelten. Da die anderen Bedingungen des Satzes 1 erfüllt sind, folgt Satz 2 aus Satz 1.

§ 3. Das Gleichheitsproblem

Wir werden hier das Funktionalgleichungssystem

$$M_\varphi(x_i)_{f_i} = M_\psi(x_i)_{g_i} \tag{14}$$

lösen, wo $\varphi, \psi \in D(I)$, $f_j, g_j \in Q(I)$ $(j=1, 2, \ldots)$, $x_i \in I$ $(n=2, 3, \ldots)$. Wir setzen ferner voraus, daß

(A_1) die Reihe $\sum\limits_{j=1}^{\infty} g_j(x)$ für $x \in I$ divergent ist,

(B_1) die (endliche) Grenzfunktion

$$\lim_{m\to\infty} \frac{\sum\limits_{j=1}^{m} f_j(t)}{\sum\limits_{j=1}^{m} g_j(t)} = G(t) \qquad (t \in I)$$

existiert. Dann gilt der

Satz 3. *Die Gleichheit*

$$M_\varphi(x_i)_{f_i} = M_\psi(x_i)_{g_i} \tag{14}$$

besteht bei beliebigen $x_i \in I$ $(n=2, 3, \ldots)$ *dann und nur dann, wenn*

$$\psi(x) = \frac{a\varphi(x)+b}{c\varphi(x)+d} \tag{15}$$

und

$$g_i(x) = kf_i(x)\big(c\varphi(x)+d\big) \qquad (i=1, 2, \ldots), \tag{16}$$

wo a, b, c, d, k *Konstanten sind mit den Einschränkungen*

$$k(c^2+d^2)(ad-bc) \neq 0 \quad \textit{und} \tag{17}$$

$$-\frac{d}{c} \notin \varphi(I), \quad \frac{a}{c} \notin \psi(I), \quad \textit{falls} \quad c \neq 0. \tag{18}$$

Hier ist $\varphi(I)$ $\big($bzw. $\psi(I)\big)$ der Wertbereich von φ (bzw. ψ) auf I.

Bemerkung. Im Spezialfall $f_j(x)=f(x)$, $g_j(x)=g(x)$ $(j=1, 2, \ldots)$ wurde dieser Satz in [3] (unter Differenzierbarkeitsvoraussetzungen) und in [2] bewiesen.

Beweis. Aus Satz 1 folgt, daß (14) dann und nur dann gilt, wenn die Gleichungen

$$f_i(u)\frac{\varphi(u)-\varphi(t)}{\varphi'(t)} = g_i(u)\frac{\psi(u)-\psi(t)}{\psi'(t)}G(t) \qquad (u, t \in I, \ i=1, 2, \ldots) \tag{19}$$

erfüllt sind. Setzen wir in (19) $t = t_0 \in I$, so erhalten wir

$$f_i(u)\Phi(u) = g\ (u)\Psi(u)G(t_0) \tag{20}$$

mit

$$\Phi(u)=\frac{\varphi(u)-\varphi(t_0)}{\varphi'(t_0)},\quad \Psi(u)=\frac{\psi(u)-\psi(t_0)}{\psi'(t_0)}. \tag{21}$$

Auf Grund von (20) wird

$$G(t)\Phi(t)=\lim_{m\to\infty}\frac{\sum_{j=1}^{m}f_j(t)\Phi(t)}{\sum_{j=1}^{m}g_j(t)}=\lim_{m\to\infty}\frac{\sum_{j=1}^{m}g_j(t)\Psi(t)G(t_0)}{\sum_{j=1}^{m}g_j(t)}=\Psi(t)G(t_0). \tag{22}$$

$\big(G(t_0)\neq 0$, da sonst wegen (19) $\varphi(u)\equiv\varphi(t_0)$ was unmöglich ist.$\big)$ Mit Hilfe von (20), (22) ergibt sich aus (19)

$$\frac{\Phi(u)-\Phi(t)}{\Phi'(t)}\Phi(t)\Psi(u)=\frac{\Psi(u)-\Psi(t)}{\Psi'(t)}\Psi(t)\Phi(u)\qquad (u,t\in I). \tag{23}$$

Aus (23) folgt für $u,t\neq t_0$

$$\frac{\Phi(u)-\Phi(t)}{\Phi'(t)}\,\frac{\Phi(t)}{\Phi(u)}=\frac{\Psi(u)-\Psi(t)}{\Psi'(t)}\,\frac{\Psi(t)}{\Psi(u)}.$$

Setzen wir hier $t=t_1\in I$, $t_1\neq t_0$, so wird wegen (21)

$$\psi(u)=\frac{a\varphi(u)+b}{c\varphi(u)+d}\qquad (u\neq t_0), \tag{24}$$

wo die Konstanten a, b, c, d durch $\varphi(t_i)$, $\varphi'(t_i)$, $\psi(t_i)$, $\psi'(t_i)$ $(i=0, 1)$ ausgedrückt werden können. Hier muß

$$c^2+d^2>0,\quad ad-bc\neq 0 \tag{25}$$

gelten, da sonst $\psi(u)$ (eventuell außer $u=t_0$) nirgends endlich bzw. konstant wäre, was unmöglich ist. Auf Grund der Stetigkeit gilt (24) auch für $u=t_0$. Damit ist die Richtigkeit der Formel (15) bewiesen. Im Falle $c\neq 0$ zeigt die Umschreibung

$$\psi(x)-\frac{a}{c}=-\frac{ad-bc}{c^2\varphi(x)+dc}$$

von (15), daß die Bedingung (18) gelten muß.

Setzen wir (15) in (19) zurück, so erhalten wir

$$\frac{g_i(u)}{f_i(u)}\,\frac{1}{c\varphi(u)+d}=\frac{1}{G(t)\big(c\varphi(t)+d\big)}\quad\text{für}\quad u\neq t. \tag{26}$$

Nehmen wir $t=t_0, t_1$ in (26), so ergibt sich

$$\frac{g_i(u)}{f_i(u)}\,\frac{1}{c\varphi(u)+d}=\text{konstant}=k\neq 0\qquad (u\in I;\ i=1,2,\ldots),$$

d.h. (16) und $\big($wegen (25)$\big)$ (17) sind erfüllt.

Gelten (15), (16), (17), (18), so sieht man leicht, daß die Gleichungen (19) auch gelten.

Damit haben wir den Satz 3 bewiesen.

§ 4. Quasihomogene Mittelwerte

Es sei E eine beliebige Menge und

$$D_E(I) = \{\Omega | \Omega: \quad E \times I \to I\}.$$

Der Mittelwert $M_\varphi(x_i)_{f_i}$ heißt quasihomogen bezüglich der Funktionenfolge $\Omega_i \in D_E(I)$ $(i=0, 1, 2, \ldots)$, falls

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n} f_i(\Omega_i(t, x_i))\varphi(\Omega_0(t, x_i))}{\sum_{i=1}^{n} f_i(\Omega_i(t, x_i))}\right) = \Omega_0\left(t, \varphi^{-1}\left(\frac{\sum_{i=1}^{n} f_i(x_i)\varphi(x_i)}{\sum_{i=1}^{n} f_i(x_i)}\right)\right) \tag{27}$$

für alle $t \in E$, $x_i \in I$; $i=1, 2, \ldots, n$; $n=2, 3, \ldots$ erfüllt ist.

Im folgenden Satz werden wir voraussetzen, daß für alle festen $t \in E$, $\Omega_0(t, \cdot) \in D(I)$,

(A$_2$) die Reihe $\sum_{j=1}^{\infty} f_j(x)$ $(x \in I)$ divergent ist,

(B$_2$) die (endliche) Grenzfunktion

$$\lim_{m\to\infty} \frac{\sum_{j=1}^{m} f_j(\Omega_j(t, x))}{\sum_{j=1}^{m} f_j(x)} = F(t, x)$$

für $t \in E$, $x \in I$ existiert.

Satz 4. *Der Mittelwert $M_\varphi(x_i)_{f_i}$ ist dann und nur dann quasihomogen bezüglich der Funktionenfolge $\Omega_i \in D_E(I)$ $(i=0, 1, 2, \ldots)$, wenn die Funktionen φ, f_i dem Funktionalgleichungssystem*

$$\left.\begin{aligned}
\varphi(\Omega_0(t, x)) &= \frac{a(t)\varphi(x)+b(t)}{c(t)\varphi(x)+d(t)},\\
f_i(\Omega_i(t, x)) &= k(t)f_i(x)\big(c(t)\varphi(x)+d(t)\big) \qquad (i=1, 2, \ldots)\\
&k(t)\big(c(t)^2+d(t)^2\big)\big(a(t)d(t)-b(t)c(t)\big) \neq 0,\\
&-\frac{d(t)}{c(t)} \notin \varphi(I), \quad \frac{a(t)}{c(t)} \notin \varphi(I) \quad \textit{falls} \quad c(t) \neq 0
\end{aligned}\right\} \tag{28}$$

genügen.

Beweis. Führen wir bei fixem $t \in E$ die Bezeichnungen

$$\psi(x) = \varphi(\Omega_0(t, x)), \quad g_i(x) = f_i(\Omega_i(t, x)) \qquad (i = 1, 2, \ldots) \tag{29}$$

ein, so erhalten wir aus (27)

$$M_\psi(x_i)_{g_i} = M_\varphi(x_i)_{f_i}. \tag{30}$$

Laut Satz 3 besteht (30) genau dann, wenn

$$\begin{aligned} &\psi(x) = \frac{a\varphi(x)+b}{c\varphi(x)+d} \\ &g_i(x) = kf_i(x)\big(c\varphi(x)+d\big) \qquad (i=1, 2, \ldots) \\ &\left(k(c^2+d^2)(ad-bc) \neq 0, \quad -\frac{d}{c} \notin \varphi(I), \quad \frac{a}{c} \notin \psi(I) \text{ falls } c \neq 0\right) \end{aligned} \tag{31}$$

gilt, wo die „Konstanten" a, b, c, d und k noch von dem bisher fix gehaltenen t abhängen. Mit (29) geht so (31) in (28) über, w. z. b. w.

Es seien jetzt $\Omega_0 = tx$, $\Omega_i = t^{r_i}x$ $(r_i \neq 0)$ $\big(i = 1, 2, \ldots;\ t, x \in (0, \infty) = I\big)$ und *wir setzen voraus, daß die Funktionen* $f_j \in Q(I)$ *stetig sind und ferner die Bedingungen* (A_2), (B_2) $\big($*mit den obigen* Ω_i $(i=0, 1, 2, \ldots)\big)$ *erfüllen.* Dann gilt der folgende Satz (vgl. [2] Satz 3, wo die Homogenitätsgleichung $M_\varphi(tx_i)_f = tM_\varphi(x_i)_f$ gelöst wurde).

Satz 5. *Der Mittelwert* $M_\varphi(x_i)_{f_i}$ *ist genau dann quasihomogen bezüglich der Funktionen* tx, $t^{r_i}x$ $\big(r_i \neq 0;\ i = 1, 2, \ldots;\ t, x \in (0, \infty) = I\big)$ *wenn* $M_\varphi(x_i)_{f_i}$ *von der Gestalt*

$$M_\varphi(x_i)_{f_i} = \exp\left(\frac{\sum_{i=1}^{n} k_i x_i^{p/r_i} \ln x_i}{\sum_{i=1}^{n} k_i x_i^{p/r_i}}\right) \tag{32}$$

oder von der Gestalt

$$M_\varphi(x_i)_{f_i} = \left(\frac{\sum_{i=1}^{n} k_i x_i^{p/r_i} x_i^a}{\sum_{i=1}^{n} k_i x_i^{p/r_i}}\right)^{1/a} \tag{33}$$

ist, wo $a \neq 0$, p, $k_i > 0$ $(i = 1, 2, \ldots)$ *Konstanten sind.*

Beweis. Nach dem Satz 4 genügen die Funktionen dem Gleichungssystem

$$\varphi(tx) = \frac{a(t)\varphi(x)+b(t)}{c(t)\varphi(x)+d(t)}, \tag{34}$$

$$f_i(t^{r_i}x) = k(t)f_i(x)\big(c(t)\varphi(x)+d(t)\big) \qquad (i = 1, 2, \ldots) \tag{35}$$

und den Nebenbedingungen

$$k(t)\big(c(t)^2+d(t)^2\big)\big(a(t)d(t)-b(t)c(t)\big) \neq 0, \tag{36}$$
$$-\frac{d(t)}{c(t)} \notin \varphi(I), \quad \frac{a(t)}{c(t)} \notin \varphi(I) \quad \text{falls} \quad c(t) \neq 0.$$

I. Betrachten wir erstens den Fall $c(t) \equiv 0$. Dann erhalten wir aus (34), (36) die Gleichung

$$\varphi(tx) = A(t)\varphi(x) + B(t) \quad \left(x, t \in (0, \infty), \quad A(t) = \frac{a(t)}{d(t)} \neq 0, \quad B(t) = \frac{b(t)}{d(t)}\right). \tag{37}$$

Eine einfache Rechnung zeigt, daß die Funktion $\Phi(x) = \varphi(x) - \varphi(1)$ der Funktionalgleichung

$$\Phi(tx) = \alpha\Phi(t)\Phi(x) + \Phi(t) + \Phi(x) \tag{38}$$

genügt. Wegen der Differenzierbarkeit von φ sind die Lösungen der Gleichung (38) die Funktionen

$$\Phi(x) = a \ln t \quad (\alpha = 0), \qquad \Phi(x) = \frac{x^a - 1}{\alpha} \quad (\alpha \neq 0)$$

(siehe [1], Seiten 59—61). Daraus folgt, daß

$$\varphi(x) = a \ln x + b \quad \text{oder} \quad \varphi(x) = \frac{1}{\alpha} x^a + b. \tag{39}$$

Mit den Bezeichnungen $t^{r_i} = s$, $k(s^{1/r_i})d(s^{1/r_i}) = D_i(s)$ folgt aus (35)

$$f_i(sx) = D_i(s) f_i(x)$$

und wegen der Stetigkeit von f_i wird $f_i(x) = k_i x^{b_i}$ $k_i > 0$. Weil $f_i(t^{r_i}x)/f_i(x)$ von i nicht abhängt, so muß $b_i r_i = p$ (konstant) sein und damit ist

$$f_i(x) = k_i x^{p/r_i} \qquad (i = 1, 2, \ldots). \tag{40}$$

Mit den Funktionen (39), (40) erhalten wir die Mittelwerte (32) und (33).

II. Zweitens untersuchen wir den Fall $c(t) \not\equiv 0$. Mit Hilfe der Bezeichnungen

$$t^{r_i} = s, \quad k(s^{1/r_i})c(s^{1/r_i}) = C_i(s), \quad k(s^{1/r_i})d(s^{1/r_i}) = D_i(s)$$

erhalten wir aus (35)

$$f_i(sx) = C_i(s) f_i(x)\varphi(x) + D_i(s) f_i(x) \qquad (s, x \in (0, \infty);\ i = 1, 2, \ldots). \tag{41}$$

Das Einsetzen $x = 1$ zeigt, daß $D_i(s) = \dfrac{f_i(s)}{f_i(1)} - C_i(s)\varphi(1)$

gilt. Setzen wir diese Formel in (41) zurück, dann wird

$$F_i(sx) = C_i(s)F_i(x)(\varphi(x)-\varphi(1))+F_i(s)F_i(x) \quad \left(F_i(x) = \frac{f_i(x)}{f_i(1)}\right).$$

Wegen der Gleichheit $F_i(sx) = F_i(xs)$ ergibt sich

$$C_i(s)F_i(x)(\varphi(x)-\varphi(1)) = C_i(x)F_i(s)(\varphi(s)-\varphi(1)),$$

d.h. mit $x=2$

$$C_i(s) = A_i F_i(s)(\varphi(s)-\varphi(1)). \tag{42}$$

Hier sind die Konstanten A_i von Null verschieden, da sonst $C_i(s)\equiv 0$ und $c(t)\equiv 0$ wäre. Mit Hilfe der Formel (42) erhalten wir die Gleichung

$$F_i(sx) = F_i(s)F_i(x)+A_i G_i(s)G_i(x), \tag{43}$$

wo $G_i(x) = F_i(x)(\varphi(x)-\varphi(1))$ ist.

Die stetigen Lösungen von (43) können mit wohlbekannten Methoden (siehe [1], S. 196—201) bestimmt werden. Diese Lösungen sind

$$F_i(x) = (\alpha_i x^{a_i}+\beta_i)x^{b_i}, \quad G_i(x) = (\gamma_i x^{a_i}+\delta_i)x^{b_i}; \tag{44}$$

$$F_i(x) = (\alpha_i \ln x+\beta_i)x^{b_i}, \quad G_i(x) = (\gamma_i \ln x+\delta_i)x^{b_i}; \tag{45}$$

$$F_i(x) = (\alpha_i \sin a_i \ln x+\beta_i \cos a_i \ln x)x^{b_i}, \tag{46}$$

$$G_i(x) = (\gamma_i \sin a_i \ln x+\delta_i \cos a_i \ln x)x^{b_i};$$

wo die Konstanten a_i, b_i, α_i, β_i, γ_i, δ_i noch gewisse Relationen erfüllen. Eine einfache Rechnung zeigt, daß die Funktionen (45), (46) der Bedingungen

$$F_i(1)=1, \quad G_i(1)=0, \quad F_i(x)>0, \quad G_i(x)\not\equiv 0 \quad \text{für} \quad x\in(0,\infty), \quad i=1,2,\ldots \tag{47}$$

und der Gleichung (43) gleichzeitig nicht genügen. Aus (44) folgen nach (43), (47) die Relationen $\beta_i = 1-\alpha_i$, $\delta_i = -\gamma_i$, $1\geqq\alpha_i>0$, $\gamma_i\neq 0$, $\alpha_i^2-\alpha_i+A_i\gamma_i^2 = 0$. Wegen $\varphi(x)-\varphi(1) = G_i(x)/F_i(x)$ hängt $G_i(x)/F_i(x)$ nicht von i ab. Dies ist nur dann möglich, wenn $a_i = a\neq 0$, $1\geqq\alpha_i=\alpha>0$, $\gamma_i=\alpha\neq 0$ $(i=1,2,\ldots)$ gilt. Wir erhalten also

$$\varphi(x) = \frac{(\gamma+\alpha\varphi(1))x^a+(1-\alpha)\varphi(1)-\gamma}{\alpha x^a+1-\alpha}, \tag{48}$$

$$f_i(x) = f_i(1)x^{b_i}(\alpha x^a+1-\alpha) \qquad (i=1,2,\ldots).$$

Da laut Satz 3 die Mittelwerte $M_\varphi(x_i)_{f_i}$ gegenüber einer Transformation der Gestalt (15), (16) invariant sind, erzeugen

$$\tilde\varphi(x) = x^a, \quad \tilde f_i(x) = k_i x^{b_i} \qquad (k_i = f_i(1),\ i=1,2,\ldots) \tag{49}$$

denselben Mittelwert, wie die Funktionen (48). Auf Grund von (35) hängt $f_i(t^{r_i}x)/f_i(x)$ nicht von i ab, also muß $b_i r_i = p$ (konstant) bzw. $b_i = p/r_i$ sein. Mit den Funktionen (49) bekommen wir wieder den Mittelwert (33). Damit haben wir den Satz 5 vollständig bewiesen.

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Generalization of a theorem of A. and C. Rényi on periodic functions

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In memory of Alfred and Catherina Rényi

A. and C. RÉNYI proved [4]

Theorem A. *If $f(z)$ is an entire function and $p(z)$ is a polynomial of degree $n \geqq 3$, then $f[p(z)]$ can not be periodic.*

We prove now the following generalization to meromorphic function:

Theorem. *Let $f(z)$ be a non-constant meromorphic function and let $p(z)$ be a polynomial of degree n. The function*

$$F(z)=f[p(z)]$$

can not be periodic unless n has one of the values $1, 2, 3, 4, 6$.

If $n=1$, then $F(z)$ can be any periodic, meromorphic function. If $n=2$, then $F(z)$ is obtained by simple changes of variable from an even periodic function. If $n \geqq 3$ then F is an elliptic function and $F(z) = g[(z+\alpha)^n]$ for a suitable meromorphic g and complex α.

Lemma. *Let*

$$p(z) = az^n + bz^{n-\nu} + \cdots \qquad (\nu \geqq 2)$$

be a polynomial of degree n. If $|z|$ is sufficiently large ($|z|>r_0$), then the roots ς of

$$p(\varsigma)=p(z) \qquad (|z|>r_0)$$

are given by

$$\varsigma = \varrho^k z + O\left(\frac{1}{|z|}\right), \qquad (k=1, 2, \ldots, n),$$

where

$$\varrho = e^{2\pi i/n}.$$

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Proof (Lemma): a simple application of the implicit function theorem to the equation

$$\left(p(\varsigma)\right)^{-\frac{1}{n}} = \varrho^{-k}\left(p(z)\right)^{-\frac{1}{n}}$$

regarding $\frac{1}{z}$ and $\frac{1}{\varsigma}$ as the basic variables.

Proof (Theorem). For $n=1$ there is nothing to prove. For $n=2$ we have, completing the square

$$f\left(p(z)\right) = f(az^2+bz+c) = f\left[a\left(z+\frac{b}{2a}\right)^2+c-(b^2/4a)\right]$$

and $F(z)$ is an even function of $z+b/2a$.

Suppose now that $n>2$ and that F is periodic. By a simple shift of origin in the z-plane we may assume

$$p(z) = az^n+bz^{n-\nu}+\cdots \qquad (\nu\geqq 2).$$

By replacing z by γz we may also suppose $F(z) = F(z+1)$. Choose z quite arbitrarily For a sufficiently large integer m the equation

$$p(\varsigma) = p(z+m)$$

has a solution

(1) $$\varsigma = \varrho(z+m)+o(1) \qquad (m\to\infty).$$

Also, if m is sufficiently large, $|\varsigma+m'|$ will be greater than r_0 (of the Lemma) for every integer m'.

From the properties of F

(2) $$F(\varsigma)=F(z).$$

Again, with ς as just defined

$$p(\varsigma') = p(\varsigma+m')$$

has a root

$$\varsigma' = \varrho(\varsigma+m')+o(1) = \varrho^2 z+\varrho^2 m+\varrho m'+o(1). \qquad (m\to\infty)$$

Also

$$F(\varsigma'+m) = F(\varsigma') = F(\varsigma) = F(z).$$

i.e. for given z the equation

(3) $$F(w)=F(z)$$

has solutions

(4) $$w = \varrho^2 z+\varrho(\varrho m+m'+\varrho^{-1}m)+o(1) \qquad (|m|>M_0,\ m' \text{ arbitrary}).$$

Now $\varrho m+\varrho^{-1}m+m' = \left(2\cos\frac{2\pi}{n}\right)m+m'$.

If $2\cos\frac{2\pi}{n}$ is irrational then $\left(2\cos\frac{2\pi}{n}\right)m+m'$ can be made arbitrarily close to any real number ξ for some arbitrarily large integer m and corresponding suitable m'. This means

$$F(\varrho^2 z+\varrho\xi) \equiv F(z) \qquad (-\infty<\xi<\infty)$$

and so F must be a constant, and the same is true of f.

If $\alpha = \cos\frac{2\pi}{n}$ is rational, then the primitive n^{th} root of unity ϱ satisfies $\varrho^2-2\alpha\varrho+1 = 0$.

But the primitive n^{th} roots of unity obey an irreducible equation of degree $\varphi(n)$, $g(\varrho)=0$; $g(\varrho)$ must divide $\varrho^2-2\alpha\varrho+1$, so that $\varphi(n)=1$ or $\varphi(n)=2$.

We have

$$\varphi(n) = n\prod_{p/n}\left(1-\frac{1}{p}\right) \geqq \prod(p-1).$$

If $\varphi(n)\leqq 2$, the only possible prime factors of n are 2 and 3 and it is now immediate that $n=3$, 4 or 6.

If $2\cos\frac{2\pi}{n}$ is rational, we can find arbitrarily large m and corresponding m' so that

$$2\cos\frac{2\pi}{n}m+m' = 0.$$

Choosing m and m' in this way and letting $m\to\infty$ we find from (3) and (4)

$$F(\varrho^2 z)=F(z).$$

In the same way, making

$$2\cos\frac{2\pi}{n}m+m' = 1, \quad F(\varrho^2 z+\varrho) = F(z) = F(\varrho^2 z).$$

Therefore F has period ϱ and F is a meromorphic function with the periods 1 and ϱ, i.e., an elliptic function. Also, by (1) and (2)

$$F\big(\varrho z+\varrho m+o(1)\big) = F\big(\varrho z+o(1)\big) = F(z).$$

In the limit $m\to\infty$

$$F(\varrho z)=F(z).$$

This shows that F is a function of z^n only and the Theorem is proved.

This result proves a conjecture in [1] and resolves problems raised in [2] and [3].

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The value distribution of composite entire functions

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1. If the entire function $F(z)$ is expressible in the form $f \circ g(z)$, where f and g are transcendental entire functions, it is called composite; otherwise $F(z)$ is said to be pseudo-prime. OZAWA [3] proved various results about the value distribution of composite entire functions, including the following:

If $F(z)$ is entire and of finite order and if there exists a constant A such that $F(z)=A$ has only real roots, then $F(z)$ is not composite.

Thus a composite entire function $F(z)$ of finite order has none of its A-values distributed entirely on a line and, a fortiori none is distributed on a ray. One can strengthen this last statement and assert that there is no direction which is the sole limiting direction of the A-points:

Theorem 1. *If $F(z)$ is an entire function of finite order and there exist complex A and real α such that for any $\delta > 0$ all but a finite number of roots of $F(z)=A$ lie in the angle $|\arg z-\alpha| < \delta$, then $F(z)$ is pseudo-prime.*

In Section 3 similar arguments to those used in the proof of Theorem 1 are applied to a question of iteration theory.

2. Proof of Theorem 1. (i) Without loss of generality, we may suppose $\alpha=\pi$. Suppose $F(z)$ satisfies the conditions of the theorem and that, nevertheless, $F=f(g)$, f and g are transcendental. Then by a result of PÓLYA [4], f has zero order and g has finite order (less than that of F).

Now $f(w)=A$ has an infinity of solutions $w=w_1, w_2, \ldots, w_n, \ldots$ and $|w_n| \to \infty$. For any $\delta > 0$, the roots of $g(z)=w_n$ $(n > n_0)$ all lie in the angle $A(\delta)$: $|\arg z-\pi| < \delta$ and so $g(z)$ omits the values w_n in $B(\pi-\delta)$: $|\arg z| \leqq \pi-\delta$.

BIEBERBACH [2] has shown that if the entire function $h(z)$ takes two different finite values at most finitely often in an angle of aperture $\alpha\pi$, then in every smaller angle

$$|f(z)|=O\{\exp(K|z|^{1/\alpha})\}$$

for a suitable constant K.

We deduce that $g(z)$ is of order $\leqq \frac{1}{2}\pi/(\pi-\delta)$ in $B(\pi-2\delta)$. Since $g(z)$ is of some finite order, say ϱ, in the whole plane, in particular in $A(2\delta)$ of aperture 4δ, δ arbitrary, it follows from the Phragmén—Lindelöf principle that $\varrho \leqq \frac{1}{2}$.

(ii) Choose $w_k \neq g(0)$ and $0<\delta<\pi/16$. Then $g(z)$ may be expressed

$$g(z)-w_k = \lambda \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right) = P(z) \prod_{n=n_0}^{\infty}\left(1-\frac{z}{z_n}\right),$$

$0 \neq \lambda$ constant, $P(z)$ polynomial. Since $f(g(z_n))=A$ we may assume that for given $\delta>0$, $z_n \in A(\delta)$ when $n \geqq n_0$.

For $z \in B\left(\frac{\pi}{2}-\delta\right)$: $|\arg z| < \frac{\pi}{2}-\delta$ and for $n \geqq n_0$ we then have $|\arg(-z/z_n)| < \frac{\pi}{2}$, and so $\left|1-\frac{z}{z_n}\right| > 1$. Thus as $z \to \infty$ in $B\left(\frac{\pi}{2}-\delta\right)$, $|g(z)| \to \infty$ faster than any power of $|z|$.

(iii) Next we show that for all large enough z, ($|z|>K$), i.e. in $|\arg z|<\delta$, we have for

$$D = zg'(z)/\{g(z)-w_k\} = \sum_{n=1}^{\infty} z/(z-z_n) \tag{1}$$

that

$$|D|>4\pi\delta^{-1}, \quad |\arg D|<2\delta. \tag{2}$$

First note that the bilinear function $t = z/(z-\beta)$ maps the line joining β to $-\beta$ onto the real axis and the angle $|\arg z - \arg(-\beta)| < 2\delta$ into the region E bounded by the two circular arcs joining 0 to 1 and making angles $\pm 2\delta$ with the positive real axis at 0. Hence for $n \geqq n_0$, when $z_n \in A(\delta)$, $t = z/(z-z_n)$ maps $B(\delta)$, which belongs to $|\arg z - \arg(-z_n)| < 2\delta$, into E, and so for each $n \geqq n_0$

$$|\mathrm{Im}\, z/(z-z_n)| \leqq \tan(2\delta)\cdot \mathrm{Re}\, z/(z-z_k) \quad \text{in} \quad B(\delta). \tag{3}$$

Since, for each fixed n, $z/(z-z_n) \to 1$ as $z \to \infty$, one has for all $z \in B(\delta)$ with sufficiently large $|z|$, that (3) holds for all n. Hence, from (1),

$$|D| \geqq \mathrm{Re}\, D > 4\pi\delta^{-1},$$

and

$$|\mathrm{Im}\, D| < (\tan 2\delta)\cdot \mathrm{Re}\, D, \quad |\arg D| < 2\delta$$

for $z \in B(\delta)$, $|z|>K$, say.

(iv) Choose w_n, $n>n_0$, such that $f(w_n)=A$ and so large that $|g(z)-w_k| < |w_n-w_k|$ for $|z| \leqq K$, where K is the constant which occurs in (iii). The component C of the set $\{z\colon |g(z)-w_k| < |w_n-w_k|\}$, which contains the origin, has a bounded intersection with $B(\delta)$ and this intersection contains $B(\delta) \cap \{|z| \leqq K\}$. Then the boundary of $C \cap B(\delta)$ contains an arc of a level curve γ of $g(z)-w_k$ which joins a point of $\arg z = -\delta$ to a point of $\arg z = \delta$ and lies in $|z|>K$. On γ one has (1) and (2) of (iii). Hence the arc γ contains no zeros of $g'(z)$. If, moreover, an increment δz on γ corresponds to an increment δw on $|w-w_k| = |w_n-w_k|$ under $w=g(z)$, then

$$\text{(4)} \qquad \frac{\delta w}{(w-w_k)} = \frac{\delta z}{z}\cdot\frac{zg'(z)}{g(z)-w_k}\{1+o(\delta z)\},$$

so that

$$\arg\left(\frac{\delta z}{z}\right)-\frac{\pi}{2} = \arg\left(\frac{\delta z}{z}\cdot\frac{w-w_k}{\delta w}\right) = -\arg\frac{zg'(z)}{(g(z)-w_k)}\{1+o(\delta z)\},$$

and by (2) $\left|\arg\left\{\frac{zg'(z)}{g(z)-w_k}\right\}\right| < 2\delta < \frac{\pi}{8}$, so the arc γ can be expressed as $z=r(\theta)e^{i\theta}$, $-\delta \leqq \theta \leqq \delta$.

Putting $w-w_k = |w_n-w_k|e^{i\varphi}$, we have in (4):

$$i\delta\varphi\{1+o(\delta\theta)\} = \left\{\frac{\delta r}{r}+i\cdot\delta\theta\right\}\left\{\frac{zg'(z)}{g(z)-w_k}\right\}\{1+o(\delta\theta)\},$$

whence

$$\left|\frac{\partial\varphi}{\partial\theta}\right| \geqq \left|\frac{zg'(z)}{g(z)-w_k}\right| > 4\pi\delta^{-1}, \text{ by (2).}$$

As z traverses γ in the direction of increasing θ, w traverses the circle Γ: $|w-w_k| = |w_n-w_k|$ in the positive direction and φ increases by at least $4\pi\delta^{-1}\cdot 2\delta = 8\pi$. Thus w traverses the whole of Γ and in particular $g(z)=w=w_n$ for some point $z\in\gamma\subset B(\delta)$. But this contradicts the fact, established in (i), that $g(z)=w_n, n>n_0$, has no roots outside $A(\delta)$. Thus the assumption that $F(z)$ is composite must be false.

3. A related question in iteration theory. Let $f(z)$ be an entire function and $f_1(z)=f(z)$, $f_2(z)=f(f(z)), \ldots, f_n(z), \ldots$ be its sequence of iterates. Regarding the Fatou set $\mathfrak{F}(f)$ of those points of the complex plane where $\{f_n(z)\}$ does not form a normal family, it was shown in [1] that *if $f(z)$ is entire and transcendental, then $\mathfrak{F}(f)$ cannot be contained in any finite set of lines* but on the other hand, *for any constant $A>0$ there exists an entire transcendental function for which $\mathfrak{F}(f)$ is contained in the region $\{|\operatorname{Im} z|<A, \operatorname{Re} z>0\}$.*

The function used to show this last result was of infinite order. In fact, using the arguments of Section 2 we can show:

Theorem 2. *If f is entire transcendental and for every $\delta>0$ the set $\mathfrak{F}(f)-$ $-\{z, (\arg z)<\delta\}$ is bounded, then f is of infinite order.*

Proof. Suppose f satisfies the hypotheses of the theorem, but is of finite order. $\mathfrak{F}(f)$ has the properties (cf. [1]):

(i) $\mathfrak{F}(f)$ is non-empty and perfect,
(ii) If $f(z)=\alpha\in\mathfrak{F}$, then $z\in\mathfrak{F}$.

We take two different values α, β in $\mathfrak{F}(f)$ which are not Picard exceptional for $f(z)$. The solutions of $f(z)=\alpha, \beta$ lie in $\mathfrak{F}$ and so, with finitely many exceptions in $|\arg z|<\delta$. Noting that $\delta>0$ is arbitrary and proceeding as in § 2 (i), we see that $f(z)$ has order at most $\frac{1}{2}$.

The method of Section 2 (ii)—(iv) then shows that in the angle B: $|(\arg z)-\pi|<\delta$ obtained from $|\arg z|<\delta$ by reflection in the origin, $f(z)$ takes all arbitrarily large values, in particular large values z_n for which $f(z_n)=\alpha$, i.e. values for which $z_n\in\mathfrak{F}$. If $f(t_n)=z_n$, $t_n\in B$, we have $t_n\in\mathfrak{F}$, since $z_n\in\mathfrak{F}$. Taking a sequence $z_n\in\mathfrak{F}$ for which $|z_n|\to\infty$, we have $|t_n|\to\infty$ and hence $\mathfrak{F}\cap B$ is unbounded or $\mathfrak{F}-\{z, |\arg z|<\delta\}$ is unbounded, against the assumptions of the theorem. Hence f must be of finite order.

In Theorem 2 the transcendence of f is essential. Polynomials have bounded $\mathfrak{F}$.

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Spectra of some Hausdorff operators

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In a recent paper [2] A. BROWN, P. HALMOS, and A. L. SHIELDS investigated the Cesàro matrix of order 1 and its continuous analogoues as operators over the Hilbert spaces l^2, $L^2[0, 1]$, and $L^2[0, \infty)$. In this paper I investigate similar properties for a class of totally regular Hausdorff matrices and their continuous analogues over the spaces l^p, $L^p[0, 1]$, and $L^p[0, \infty)$ for $p > 1$.

1. Discrete methods

Let $\mu = \{\mu_n\}$ be a sequence, Δ the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^n \mu_k = \Delta(\Delta^{n-1}\mu_k)$; $k = 0, 1, 2, \ldots$; $n = 1, 2, 3, \ldots$. A Hausdorff matrix H is defined by $h_{nk} = \binom{n}{k} \Delta^{n-k}\mu_k$ for $k \leqq n$, $h_{nk} = 0$ for $k > n$. For a regular matrix (i.e., one that preserves limits for convergent sequences) we have the representation

$$\mu_n = \int_0^1 x^n \, dq(x) \qquad (n = 0, 1, 2, \ldots),$$

where $q \in BV[0, 1]$, $q(0+) = q(0) = 0$, $q(1) = 1$, and $q(u) = [q(u+0) + q(u-0)]/2$ for $0 < u < 1$. If in addition q is nonnegative and nondecreasing over $[0, 1]$, then H is called totally regular. For other properties of Hausdorff matrices the reader may consult [4, XI].

First we shall establish some properties for all totally regular Hausdorff matrices that are defined and bounded on l^p, and then examine some of the specific methods. Let $\|H\|_p$ denote the l^p norm of such a matrix H.

Theorem 1. *Set* $H(p) = \int_0^1 x^{-1/p} \, dq(x)$. *Then* $\|H\|_p = H(p)$.

HARDY [3] shows that $\|Hs\|^p \leqq [H(p)]^p \|s\|^p$ for any positive sequence $s = \{s_n\} \in l^p$. His result is clearly extendable to an arbitrary sequence $s \in l^p$ by observing that $\|Hs\|_p^p = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} h_{nk} s_k \right|^p \leqq \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} h_{nk} |s_k| \right]^p$. Hence $\|H\|_p \leqq H(p)$.

To prove the converse we use the argument on pages 48 and 49 of [3] to get

$$H_n(s) > (1-\eta)^2 H(p) s_n$$

for $s_n = (n+1)^{-\omega}$, $\omega = \dfrac{1}{p} + \varepsilon$, $0 < \varepsilon < \dfrac{1}{p}$, $\eta > 0$ and arbitrary. This result leads to $\|H\|_p \geqq H(p)$.

Some of the well-known Hausdorff matrices which are bounded operators over l^p are the Cesàro, Hölder, Euler, gamma, and generalized-Cesàro. These are listed below along with their generating sequences and mass functions.

$$C_\alpha\colon \mu_n = \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)}; \quad q(x) = 1-(1-x)^\alpha;$$

$$H_\alpha\colon \mu_n = (n+1)^{-\alpha}; \quad q(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (\log(1/t))^{\alpha-1}\,dt;$$

$$(E, r)\colon \mu_n = a^n,\ r = (1-a)/a; \quad q(x) = \begin{cases} 0, & 0 \leqq x < a < 1 \\ 1, & a \leqq x \leqq 1; \end{cases}$$

$$\Gamma_a^\alpha\colon \mu_n = \left(\frac{a}{n+a}\right)^\alpha; \quad q(x) = \frac{a^\alpha}{\Gamma(\alpha)} \int_0^x t^{a-1}(\log(1/t))^{\alpha-1}\,dt;$$

$$C_a^\alpha\colon \mu_n = \frac{\Gamma(a+\alpha)\Gamma(n+a)}{\Gamma(a)\Gamma(n+a+\alpha)}; \quad q(x) = \frac{\Gamma(a+\alpha)}{\Gamma(a)\Gamma(\alpha)} \int_0^x t^{a-1}(1-t)^{\alpha-1}\,dt.$$

From Theorem 1, with q satisfying $1/p + 1/q = 1$, the corresponding l^p-norms are:

$$\|C_\alpha\|_p = \frac{\Gamma(1+\alpha)\Gamma(1/q)}{\Gamma(\alpha+1/q)}; \quad \|H_\alpha\|_p = q^\alpha; \quad \|(E, r)\|_p = (1+r)^{1/p};$$

$$\|\Gamma_a^\alpha\|_p = \left(\frac{a}{a-1/p}\right)^\alpha \quad \|C_a^\alpha\|_p = \frac{\Gamma(a+\alpha)\Gamma(a-1/p)}{\Gamma(a+\alpha-1/p)}.$$

From the above it is clear that the operators are bounded for $\alpha > 0$, $r > 0$, $a > 1/p$.

We shall now show that Γ_a^α is not bounded for $0 < a \leqq 1/p$. If (h_{nk}) denotes the corresponding matrix, then $\Gamma_a^\alpha(e_0) = \{h_{n0}\}$, where $h_{n0} = \dfrac{\Gamma(a+1)\Gamma(n+1)}{\Gamma(n+a+1)}$, and

$$\|\Gamma_a^\alpha\|_p^p \geqq \|\Gamma_a^\alpha(e_0)\|_p^p = \sum_{n=0}^\infty (h_{n0})^p = \Gamma^p(a+1) \sum_{n=0}^\infty \left(\frac{\Gamma(n+1)}{\Gamma(n+a+1)}\right)^p.$$

Using the Gauss test one can show that the series diverges for $0 < a \leqq 1/p$. For the C_a^α matrix,

$$h_{n0} = \frac{\Gamma(n+\alpha)\Gamma(a+\alpha)}{\Gamma(\alpha)\Gamma(n+a+\alpha)},$$

and, as above, it can be shown that C_a^a is not bounded for $0 < a \leqq 1/p$.

Theorem 2. *If $H \neq I$, the point spectrum of H is empty.*

Proof. Suppose $Hf = \lambda f$ for some λ. Since H is totally regular, $\mu_n \neq 0$ for each n. Thus H is not a left zero divisor in l^p and $\lambda = 0$ is not possible. Define $g(n) = \sum_{k=0}^{n} h_{nk} f(k)$ for $f \in l^p$. Since $g(0) = \lambda f(0)$, we must have $\lambda = 1$ for any f with $f(0) \neq 0$.

Case I. Assume $f(0) \neq 0$. Then $\lambda = 1$ and we have $(H-I)f \equiv 0$. In particular $h_{10}f(0) + (h_{11}-1)f(1) = 0$; i.e.,

$$f(1) = \frac{h_{10}f(0)}{1-h_{11}}.$$

But $h_{10} = \binom{1}{0}\Delta\mu_0 = \mu_0 - \mu_1 = 1 - h_{11} > 0$. Therefore $f(1) = f(0)$ and, by induction, $f(n) = f(0)$, $n = 1, 2, \dots$. Since $f(0) \neq 0$, $f = \{f(0)\} \notin l^p$.

Case II. Assume $f(0) = 0$. Then either there exists an integer N such that $\mu_N = \lambda$ or else $\mu_n \neq \lambda$ for any n.

Case IIA. $f(0) = 0$ and $\mu_n \neq \lambda$ for any n. From the equation $h_{10}f(0) + h_{11}f(1) = \lambda f(1)$ we get $(\lambda - \mu_1)f(1) = 0$ which implies $f(1) = 0$. By induction, $f(n) = 0$; $n = 0, 1, 2, \dots$.

Case IIB. $f(0) = 0$ and $\mu_N = \lambda$ for some N. If $N = 0$, then $\lambda = 1$ and we must have $h_{10}f(0) = h_{11}f(1) = f(1)$; i.e., $(1-\mu_1)f(1) = 0$. Since $H \neq I$, $\mu_1 \neq 1$. Therefore $f(1) = 0$ and by induction, $f(n) = 0$ for $n = 2, 3, 4, \dots$.

Since $\mu_n > \mu_{n+1}$ for each r (a well-known property for totally regular Hausdorff matrices $H \neq I$), if $N > 0$ then clearly $f(0) = 0$ implies $f(1) = f(2) = \dots = f(N-1) = 0$ and $f(N)$ remains undetermined.

If $f(N) = 0$, then, as before, $f \equiv 0$.

If $f(N) \neq 0$ we shall show by induction that

$$f(N+r) = \binom{N+r}{N} f(N), \qquad r = 0, 1, 2, \dots .$$

This is trivially true for $r = 0$. Assume the induction hypothesis. Then

$$\sum_{k=0}^{N+r+1} h_{N+r+1,k} f(k) = \mu_N f(N+r+1)$$

or

$$(\mu_N-\mu_{N+r+1})f(N+r+1)=\sum_{k=N}^{N+r}h_{N+r+1,k}f(k)=\sum_{j=0}^{r}h_{N+r+1,N+j}f(N+j)=$$

$$=\sum_{j=0}^{r}\binom{N+r+1}{N+j}(\Delta^{r+1-j}\mu_{N+j})\binom{N+j}{N}f(N)=$$

$$=f(N)\binom{N+r+1}{N}\sum_{j=0}^{r}\binom{r+1}{j}\Delta^{r+1-j}\mu_{N+j}.$$

Note that $\sum_{j=0}^{r+1}\binom{r+1}{j}\Delta^{r+1-j}\mu_{N+j}$ is row $(r+1)$ of the Hausdorff matrix with generating sequence $\{\mu_{N+r}\}_{r=0}^{\infty}$.

Therefore

$$\sum_{j=0}^{r}\binom{r+1}{j}\Delta^{r+1-j}\mu_{N+j}=\mu_N-\mu_{N+r+1}\neq 0,$$

and we get $f(N+r+1)=\binom{N+r+1}{N}f(N)$. Moreover, $|f(N+r+1)|>|f(N+r)|$ so that $f\notin l^p$.

Theorem 3. *For $H\neq I$, H^*-N has a total set of proper vectors corresponding to proper values of modulus strictly less than N.*

Proof. Define a family of sequences $\beta_0,\beta_1,\beta_2,\dots$ with $\beta_n=\Delta^n e_0$, where $\Delta e_0=e_0-e_1$, $\Delta^n e_0=\Delta(\Delta^{n-1}e_0)$. The set $\{\beta_0,\beta_1,\dots\}$ is total over l^q, q the conjugate index of p. For $m>n$, $H^*\beta_n(m)=0$. For $m\leqq n$

$$H^*\beta_n(m)=\sum_{k=m}^{\infty}h^*_{mk}\beta_n(k)=\sum_{k=m}^{n}h^*_{mk}(-1)^k\binom{n}{k}=$$

$$=\sum_{k=m}^{n}\binom{k}{m}(\Delta^{k-m}\mu_m)(-1)^k\binom{n}{k}=\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k}(-1)^k\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}\mu_{m+j}=$$

$$=\sum_{r=0}^{n-m}\binom{r+m}{m}\binom{n}{r+m}(-1)^{r+m}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu_{m+j}=$$

$$=\binom{n}{m}\sum_{j=0}^{n-m}(-1)^{m+j}\mu_{m+j}\sum_{r=j}^{n-m}(-1)^r\frac{(n-m)!}{(n-m-r)!\,j!\,(r-j)!}=$$

$$=(-1)^m\binom{n}{m}\sum_{j=0}^{n-m}\binom{n-m}{j}\mu_{m+j}\sum_{s=0}^{n-m-j}(-1)^s\binom{n-m-j}{s}=(-1)^m\binom{n}{m}\mu_n=\mu_n\beta_n(m).$$

Therefore $(H^*-N)\beta_n=(\mu_n-N)\beta_n$.

Based on the knownledge of the spectrum of C_1 for $p=2$, and Theorem 3, one might conjecture that $\sigma(H)$ is the disk $\{\lambda: |\lambda-N| \leqq N\}$. Unfortunately, the conjecture is false, as the following example indicates.

From [2], $\sigma(C_1)=\{\lambda: |\lambda-1| \leqq 1\}$. Let $z = 1+e^{i\theta}$. Then $z^2 = 2(1+\cos\theta)e^{i\theta}$. Let $w = x+iy = z^2$. Then putting w in polar form yields the cardioid $r = 2(1+\cos\theta)$. Since $\sigma(C_1^2)=(\sigma(C_1))^2$, $\sigma(C_1^2)$ is the closed bounded region with the above cardioid as boundary.

There is, however, a class of totally regular Hausdorff methods H for which $\sigma(H) = \{\lambda: |\lambda-N| \leqq N\}$. This class includes the gamma methods of order 1.

Theorem 4. $\sigma(\Gamma_a^1) = \{\lambda: |\lambda-N| \leqq N\}$.

The operator $N-\Gamma_a^1-\lambda$ has moment generating sequence

$$\mu_n = c-\frac{a}{n+a},$$

where $c = N-\lambda$. Let $\varepsilon_n = 1/\mu_n$. If it can be shown that H_ε is a bounded operator over l^p for $|\lambda|>N$, then $\sigma(N-\Gamma_a^1) \subseteqq \{\lambda: |\lambda| \leqq N\}$. Hence $\sigma(\Gamma_a^1) \subseteqq \{\lambda: |\lambda-N| \leqq N\}$. Using the method of proof of (4) of [2, Theorem 2] one can show that if λ satisfies $|\lambda-N| < N$, then λ is a simple proper value of Γ_a^{1*}. The fact that the spectrum is closed completes the theorem.

We shall now show that H_ε is as required. Indeed,

$$\varepsilon_n = \frac{1}{c}\left[1+\frac{a/c}{n+a-a/c}\right].$$

Thus

$$\|H_\varepsilon\|_p \leqq \frac{1}{|c|}+\frac{a}{|c|^2}\,\|H_\delta\|_p,$$

where $\delta_n = \dfrac{1}{n+a-a/c}$, and the theorem reduces to showing that $\|H_\delta\|_p$ is finite.

Let $x \in l^p$. Then

$$\|H_\delta x\|_p = \left\{\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} h_{nk}x_k\right|^p\right\}^{1/p} \leqq \left\{\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} |h_{nk}|\,|x_k|\right]^p\right\}^{1/p}.$$

Now

$$|h_{nk}| = \binom{n}{k}|\Delta^{n-k}\mu_k| = \binom{n}{k}\left|\int_0^1 t^{k+a-a/c-1}(1-t)^{n-k}\,dt\right| \leqq$$

$$\leqq \binom{n}{k}\int_0^1 t^{k+a-\mathrm{Re}(a/c)-1}(1-t)^{n-k}\,dt.$$

Thus $\|H_\delta x\|_p \leqq H_{|\delta|}(p)\,\|x\|_p$, where

$$H_{|\delta|}(p) = \int_0^1 t^{-\frac{1}{p}+a-\mathrm{Re}(a/c)-1}\,dt,$$

provided the integral exists. It remains to show that $a-\mathrm{Re}(a/c)-1/p > 0$.

Note that $c = N-\lambda$. Thus $a-\mathrm{Re}(a/c) = a+a\left[\frac{\alpha-N}{(\alpha-N)^2+\beta^2}\right]$.

By hypothesis $|\lambda|>N$. If we let $\lambda = \alpha+i\beta$, then $|\lambda|>N$ is equivalent to $\alpha^2+\beta^2 > N^2$, which can be written in the form $(\alpha-N)^2+\beta^2 > 2N(N-\alpha)$. Hence $\frac{\alpha-N}{(\alpha-N)^2+\beta^2} > -\frac{1}{2N}$.

The proof is now complete, since $1-1/2N = 1/ap$.

2. Finite continuous methods

Theorem 5. *Let T be an integral Hausdorff transformation defined by*

$$\text{(1)} \qquad T(f)(y) = \int_0^1 f(xy)\,dq(x),$$

where q is an absolutely continuous totally regular mass function. Then for each T which is a bounded operator over $L^p[0, 1]$, $\|T\|_p = H(p)$.

Proof. From [5, p. 243], $\|Tf\|_p \leqq H(p)\|f\|_p$, with equality holding only for $f\equiv 0$ or $T\equiv I$.

To prove the reverse inequality, let $f_1(x)=x^{-\beta}$, $\beta > 1/p$. Then $f_1\in L^p[0, 1]$ and $(Tf_1)(y)=y^{-\beta}H(p)$. Therefore $\|T\|_p \geqq H(p)$.

Theorem 6. *For each $T\neq I$, $NI-T$ has a total set of proper vectors corresponding to proper values of modules strictly less than N.*

With $f_n(x)=x^n$, $n=0, 1, 2, \ldots$, the family $\{f_0, f_1, f_2, \ldots\}$ is total over $L^p[0, 1]$, and $(Tf_n)(y) = \int_0^1 (xy)^n\,dq(x) = \mu_n y^n$. Therefore $((N-T)f_n)(y) = (N-\mu_n)y^n$.

Since T^* is playing the role over $L^q[0, 1]$ that was played by H over l^p, one conjectures that T^* has empty point spectrum. The conjecture remains to be verfied general, but is true in the following special cases.

Theorem 7. *For α a positive integer, the point spectrum of H^*_α is empty.*

Proof. H_α^* has kernel

$$k^*(x,y)=\begin{cases}0, & 0<y\leqq x\\ \dfrac{1}{y\Gamma(\alpha)}(\log(y/x))^{\alpha-1}, & 0<x<y.\end{cases}$$

Suppose $H_\alpha^* g=\lambda g$ for some nonzero $g\in L^q[0,1]$, where q is the conjugate index of p. Then we have

$$\frac{1}{\Gamma(\alpha)}\int_x^1 \frac{1}{y}\left(\log\frac{y}{x}\right)^{\alpha-1} g(y)\,dy=\lambda g(x). \tag{2}$$

If $\lambda=0$, then differentiating the above gives

$$\int_x^1 \frac{g(y)}{y}\left(\log\frac{y}{x}\right)^{\alpha-2} dy\equiv 0.$$

Differentiating $\alpha-1$ more times leads to $-g(x)/x\equiv 0$ or $g\equiv 0$, a contradiction.

With $\lambda\neq 0$, (2) implies the differentiability of g. Differentiation yields

$$\lambda x g'(x)=-\frac{1}{\Gamma(\alpha-1)}\int_x^1 \frac{g(y)}{y}\left(\log\frac{y}{x}\right)^{\alpha-2} dy \tag{3}$$

Now let $w=g'(x)$, and regard g as a function of t, where $t=\log x$. Then $xg'(x)=D_t w$. Differentiating (3) $(\alpha-2)$ more times yields $\lambda D_t^\alpha w+(-1)^{\alpha+1}w=0$, which has solution

$$g(x)=\sum_{k=1}^{\alpha} A_k x^{a_k},$$

where each a_k is a root of the auxiliary equation $a^\alpha+(-1)^{\alpha+1}=0$.

From (2) and (3) it is clear that g and each of its first $\alpha-1$ derivatives vanish at $x=1$, giving rise to the system

$$\sum_{k=1}^{\alpha} A_k=0;\quad \sum_{k=1}^{\alpha} a_k(a_k-1)\dots(a_k-j)A_k=0\qquad (j=0,1,\dots,\alpha-2),$$

which is equivalent to the system

$$\sum_{k=1}^{\alpha} a_k{}^j A_k=0\qquad (j=0,1,\dots,\alpha-1).$$

This latter system has a Vandermonde determinant. Therefore each $A_k=0$ and $g\equiv 0$.

Theorem 8. *For α a positive integer, the point spectrum of C_α^* is empty.*

The method of proof is similar to that of Theorem 7. The kernel for C_α^* is

$$k^*(x,y)=\begin{cases}0, & 0<y\leqq x\\ \dfrac{\alpha}{y}\left(1-\dfrac{x}{y}\right)^{\alpha-1}, & 0>x>y.\end{cases}$$

The condition $C_\alpha^* g = \lambda g$ becomes

$$\lambda g(x) = \int_x^1 \frac{\alpha}{y}\left(1-\frac{x}{y}\right)^{\alpha-1} g(y)\,dy.$$

For $\lambda \neq 0$, the corresponding differential equation is

$$\lambda x^\alpha g^{(\alpha)}(x) - (-1)^\alpha \Gamma(\alpha+1) g(x) = 0,$$

which is of Euler-type, with solution

$$g(x) = \sum_{k=1}^{\alpha} A_k x^{a_k}.$$

As before, each A_k is zero so that $g \equiv 0$.

Theorem 9. *Γ_a^{1*} has empty point spectrum.*

The kernel for Γ_a^{1*} is

$$k^*(x, y) = \begin{cases} 0, & 0<y\leqq x \\ \dfrac{a}{y}\left(\dfrac{x}{y}\right)^{a-1}, & 0<x<y. \end{cases}$$

The condition $\Gamma_a^{1*} g = \lambda g$ leads for $\lambda \neq 0$ to the differential equation $\lambda x g'(x) = [\lambda(a-1)-a]g(x)$, which has solution $g(x) = Cx^{a-1-a/\lambda}$. Since $g(1)=0$, $C=0$, and $g \equiv 0$.

Theorem 10. $\sigma(\Gamma_a^1) = \{\lambda : |\lambda - N| \leqq N\}$.

To prove $\sigma(\Gamma_a^1) \subseteqq \{\lambda : |\lambda - N| \leqq N\}$ apply the corresponding argument of Theorem 4 to $L^p[0, 1]$.

For the opposite inclusion, suppose

$$\left(1-\frac{1}{N}\Gamma_a^1\right) f(x) = \lambda f(x).$$

The resulting differential equation has solution

$$f(x) = c_1 \exp\left[-a\left(1-1/N(1-\lambda)\right)\log x\right].$$

It is a straightforward exercise to verify that if $|\lambda|<1$, then $f \in L^p[0, 1]$. Therefore point spectrum of $\left(1-\dfrac{1}{N}\Gamma_a^1\right)$ contains the open disc $\{\lambda : |\lambda|<1\}$. Hence $\sigma(\Gamma_a^1) \supset \{\lambda : |\lambda - N| < N\}$. The proof is now complete since the spectrum is closed.

Specializing to $L^2[0, 1]$ we have the following result. Let be a continuous bounded operator over $L_2[0, 1]$ with kernel

$$k(x, y) = \begin{cases} 0, & 0<x\leqq y \\ \dfrac{1}{x} f\left(\dfrac{y}{x}\right), & 0<y<x \end{cases} \tag{4}$$

and adjoint kernel

$$k^*(x, y) = \begin{cases} 0, & 0 < y \leqq x \\ \dfrac{1}{y} f\left(\dfrac{x}{y}\right), & 0 < x < y, \end{cases}$$

where f is nonnegative and integrable over $[0, 1]$.

Theorem 11. *T^* is hyponormal.*

Proof. The kernels for TT^* and T^*T are

$$I_1 = \int_0^1 k(x, u) k^*(u, y)\, du = \frac{1}{xy} \int_0^{\min(x,y)} f\left(\frac{u}{x}\right) f\left(\frac{u}{y}\right) du$$

and

$$I_2 = \int_0^1 k^*(x, u) k(u, y)\, du = -\frac{1}{xy} \int_{\min(x,p)}^{xy} f(z/y) f(z/x)\, dz.$$

Hence

$$I_1 - I_2 = \frac{1}{xy} \int_0^{xy} f\left(\frac{z}{y}\right) f\left(\frac{z}{x}\right) dz = \int_0^1 f(wx) f(wy)\, dw.$$

For any $g \in L^2[0, 1]$,

$$((I_1 - I_2) g, g) = \int_0^1 \int_0^1 g(y) \int_0^1 f(wx) f(wy)\, dw\, dy\, \bar{g}(x)\, dx = \int_0^1 (Fg, Fg)\, du,$$

where $F(g)(w) = \int_0^1 g(y) f(wy)\, dy$.

If g is absolutely continuous then we may write $g(x) = \int_0^x h(t)\, dt$. An elementary change of variable in (1) will change h to the form in (4). Thus every totally regular integral method with absolutely continuous mass function will have its adjoint hyponormal.

3. Infinite continuous methods

Theorem 12. *Let T be a bounded linear operator over $L^p[0, \infty)$ with kernel defined by* (4). *Then* $\|T\|_p = H(p)$.

This theorem is a special case of [5, Th. 319].

Theorem 13. *For α a positive integer H_α and H_α^* have empty point spectra.*

For a proof, combine the facts that $\sigma(H_\alpha) = (\alpha(H))^\alpha$ and that $H = C_1$ has empty point spectrum. The same applies to H_α^*.

Theorem 14. *For α a positive integer C_α and C_α^* have empty point spectra.*

For each operator the proof is similar to that of Theorem 8. In each case one shows that the only solution function in the appropriate space is the zero function.

Theorem 15. *Γ_a^1 and Γ_a^{1*} have empty point spectra.*

The proofs here parallel that of Theorem 9.

Theorem 16. $\sigma(\Gamma_a^1) = \{\lambda : |\lambda - N| = N\}$.

To prove this theorem one follows the argument of [1] using

$$P(x)(t) = \frac{1}{t}\int_0^t x(s)\,a\left(\frac{s}{t}\right)^{a-1} ds$$

with

$$P_\zeta x(t) = \int_0^1 a x(st)s^{a(1-\zeta)-1}\,ds \quad \text{and} \quad Q_\zeta x(t) = \int_0^\infty a x(s)s^{a(1-\zeta)-1}\,ds$$

as the corresponding resolvents in the appropriate regions.

For $p=2$ we have the following result.

Theorem 17. *Let T be a continuous bounded operator over $L_2[0, 1]$ with kernel as in (4) with f now integrable over $[0, \infty)$. Then T^* is normal.*

The kernels corresponding to TT^* and T^*T are

$$I_1 = \int_0^\infty k(x,u)k^*(u,y)\,du = \frac{1}{xy}\int_0^{\min(x,y)} f\left(\frac{u}{x}\right)f\left(\frac{u}{y}\right)du$$

and

$$I_2 = \int_0^\infty k^*(x,u)k(u,y)\,du = \frac{1}{xy}\int_0^{\min(x,y)} f\left(\frac{z}{y}\right)f\left(\frac{z}{x}\right)dz;$$

hence $I_1 - I_2 \equiv 0$.

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Probabilistic version of Trotter's exponential product formula in Banach algebras

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1. Introduction and results

It is an elementary fact that the exponential function may be defined by the equivalent formulae

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{1}{n}x\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

not only when x is a real or complex number but also when it is a matrix with real or complex entries or a bounded operator acting on a Hilbert space or a Banach space or, even when it is an element of an abstract Banach algebra $\mathfrak{B}$ with identity I (for a definition of a Banach algebra see for instance [1]). If $\mathfrak{B}$ is not commutative then in general $\exp(x)\exp(y) \neq \exp(x+y)$. There is, however, a formula which replaces the addition law of the exponential function, namely

$$(1)\qquad \lim_{n\to\infty}\left(\exp\left(\frac{x}{n}\right)\exp\left(\frac{y}{n}\right)\right)^n = \exp(x+y)$$

and this holds regardless whether x and y commute or not. Formula (1) is capable of further generalization; see TROTTER [2]. Specifically, x and y may be unbounded operators of a certain type, namely generators of continuous one-parameter operator semi-groups. In the present paper we are not concerned with Trotter's generalization, but we shall still refer to (1) as the Trotter product formula. The symbols $x, y, \ldots, a, b, \ldots$ shall generally denote elements of the Banach algebra $\mathfrak{B}$. The norm of $x\in\mathfrak{B}$ is written $\|x\|$.

Let $\mathbf{X}=(x_1, x_2, \ldots, x_m$ be any finite sequence of elements of $\mathfrak{B}$. With any such sequence we associate the product

$$T(\mathbf{X}) = \exp\left(\frac{x_1}{m}\right)\exp\left(\frac{x_2}{m}\right)\ldots\exp\left(\frac{x_m}{m}\right)$$

which will be called its Trotter product. Note that it depends essentially on the order of the factors, i.e. on $\mathbf{X}$ as a sequence, not merely as a set. We also write, for the mean of the elements of $\mathbf{X}$, $M(\mathbf{X}) = \frac{1}{m}(x_1+x_2+\cdots+x_m)$. Using this notation we can express the Trotter product formula as follows: If $\mathbf{X}_k$ (for $k=1, 2, \ldots$) is the sequence of length $2k$ whose elements are alternatingly x and y, then

$$\lim_{k\to\infty} T(\mathbf{X}_k) = \exp\left(\lim_{k\to\infty} M(\mathbf{X}_k)\right). \tag{2}$$

We now raise the following question. Under what natural conditions on a sequence $\mathbf{X}_1, \mathbf{X}_2, \ldots$ (subject to the requirement that their lengths tend to infinity) will (2) hold?

We shall prove two theorems relevant to this question. The first theorem gives a rather general sufficient condition for an infinite sequence $\mathbf{X}_1, \mathbf{X}_2, \ldots$ to satisfy (2). The original Trotter formula is an obvious consequence of this condition. But our theorem also shows that to a certain extent the order of the factors in the Trotter product may be made subject to considerable rearrangement without destroying the validity of (2).

For any $\mathbf{X}=(x_1, x_2, \ldots, x_m)$ let us write $\varrho=\varrho(\mathbf{X})=\mathrm{Max}_{1\leqq j\leqq m}\|x_j\|$. Let π denote a partition of the sequence $(1, 2, \ldots, m)$ into successive subsequences $(1, 2, \ldots, m_1)$, $(m_1+1, m_1+2, \ldots, m_1+m_2)$, $\ldots$, $(m-m_s+1, m-m_s+2, \ldots, m)$, and let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_s$ be the corresponding subsequences of $\mathbf{X}$. For any element $g\in\mathfrak{B}$ we introduce a quantity $\delta=\delta(\mathbf{X}, \pi, g)$ which measures the closeness to which g uniformly approximates the "partial means" of $\mathbf{X}$ induced by the partition π

$$\delta(\mathbf{X}, \pi, g) = \mathrm{Max}_{1\leqq j\leqq s}\|M(\mathbf{Y}_j)-g\|. \tag{3}$$

We also define a quantity

$$\eta = \eta(\pi) = \sum_{j=1}^{s}\left(\frac{m_j}{m}\right)^2. \tag{4}$$

If $\eta_0 = \max_{1\leqq j\leqq s}\left(\frac{m_j}{m}\right)$, we have clearly

$$\eta_0^2 \leqq \eta \leqq \eta_0, \tag{5}$$

so that η is a kind of a measure for the relative fineness of π.

Theorem 1. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be an infinite sequence of finite sequences of elements of $\mathfrak{B}$. Suppose that $\varrho(\mathbf{X}_k)$ is bounded. Suppose that $g\in\mathfrak{B}$ exists and a sequence of partitions π_k of $\mathbf{X}_k$ into successive subsequences exists such that $\eta(\pi_k)\to 0$ and $\delta(\mathbf{X}_k, \pi_k, g)\to 0$ as $k\to\infty$. Then $M(\mathbf{X}_k)\to g$ and $T(\mathbf{X}_k)\to\exp(g)$ as $k\to\infty$.*

Note that the quantity $\delta(\mathbf{X}, \pi, g)$ is independent of the order of the elements x_i *within* each of the subsequences $\mathbf{Y}_j$ $(j=1, 2, \ldots, s)$. Thus our theorem formulates mathematically the intuitive notion that the order of the factors in the Trotter product $T(\mathbf{X})$ is irrelevant "locally" and only essential "in the large".

Our second Theorem deals with the following problem. Suppose we consider infinite sequences $\mathbf{Z}=(x_1, x_2, \ldots)$ whose elements are all taken from a fixed finite subset $\{a_1, a_2, \ldots, a_\sigma\}$ of $\mathfrak{B}$. With any such infinite sequence we may consider the sequence of its finite sections $\mathbf{X}_1=(x_1)$, $\mathbf{X}_2=(x_1, x_2)$, ... Is it possible to characterize the extent of the set of those infinite sequences $\mathbf{Z}$ for which the generalized Trotter product formula (2) holds?

This question leads to measure-theoretic considerations, and can be naturally formulated in probabilistic terminology.

Theorem 2. *Let $a_1, a_2, \ldots, a_\sigma \in \mathfrak{B}$, and let $p_1, p_2, \ldots, p_\sigma \geqq 0$, $\Sigma_\nu p_\nu = 1$. Suppose an infinite (random) sequence $\mathbf{Z}=(x_1, x_2, \ldots)$ is formed by choosing, independently for each $j \geqq 1$, $x_j = a_\sigma$ with probability p_σ. Let $g = p_1 a_1 + p_2 a_2 + \cdots + p_\sigma a_\sigma$. Then the probability is unity that $M(\mathbf{X}_k) \to g$ and $T(\mathbf{X}_k) \to \exp(g)$ as $k \to \infty$.*

Thus in the sense of the given probability measure defined on the set of sequences $\mathbf{Z}$, for almost every sequence the generalized Trotter product formula holds.

2. An auxiliary inequality

Both theorems derive from an elementary estimate formulated as follows:

Lemma. *Let $\mathbf{X}$ be a finite sequence of elements of $\mathfrak{B}$, π any partition of it into successive subsequences, and $g \in \mathfrak{B}$. Let $\varrho = \varrho(\mathbf{X})$, $\eta = \eta(\pi)$ and $\delta = \delta(\mathbf{X}, \pi, g)$ be defined as above. Then*

$$\|T(\mathbf{X}) - \exp(g)\| \leqq e^{\|g\|}\left(e^{\delta + \frac{1}{2}\eta\varrho^2 e^\varrho} - 2e^{-\frac{1}{2}\eta\|g\|^2} + 1\right).$$

In the proof of the Lemma we shall make use of the following facts:

(A) If $P(x_1, x_2, \ldots, x_q)$ is a polynomial or a power series with non-negative real coefficients then

$$\|P(x_1, x_2, \ldots, x_q)\| \leqq P(\|x_1\|, \|x_2\|, \ldots, \|x_q\|)$$

for all $x_j \in \mathfrak{B}$ such that the right hand side is finite.

(B) For any $t \geqq 0$, $e^t - 1 - t \leqq \frac{1}{2} t^2 e^t$.

(C) For any $t \geqq 0$, $e^{t-\frac{1}{2}t^2} \leqq 1 + t \leqq e^t$.

Let $\mathbf{Y}_j$ $(j=1, 2, \ldots, s)$ be the subsequences of $\mathbf{X}$ produced by the partition π, and let m_j be their respective length, $m = \Sigma_j m_j$. Let $T(\mathbf{X}) = y_1 y_2 \ldots y_s$ where y_j is the product of those factors in the product, taken in their proper order, which involve the elements $x_k \in \mathbf{Y}_j$. Write $y_j = 1 + \frac{m_j}{m} g + r_j$, thus defining r_j. For notational convenience we now consider $j=1$. We have

$$r_1 = \left[y_1 - 1 - \frac{m_1}{m} M(\mathbf{Y}_1)\right] + \frac{m_1}{m}[M(\mathbf{Y}_1) - g].$$

The norm of the second term is bounded by $\frac{m_1}{m}\delta$. To obtain a bound on the norm of the first term we note that if

$$y_1 - 1 - \frac{m_1}{m} M(\mathbf{Y}_1) = \exp\left(\frac{x_1}{m}\right)\exp\left(\frac{x_2}{m}\right) \dots \exp\left(\frac{x_{m_1}}{m}\right) - 1 - \frac{1}{m}(x_1 + x_2 + \dots + x_{m_1})$$

is regarded as a power series in the x_j $(j=1, 2, \dots, m_1)$ it has non-negative coefficients (the negative terms cancel!). Thus by principle (A) above we may replace x_j by $\|x_j\|$, and then taking (B) and the definition of ϱ into account we get

$$\left\| y_1 - 1 - \frac{m_1}{m} M(\mathbf{Y}_1)\right\| \leqq \frac{1}{2}\left(\frac{m_1}{m}\right)^2 \varrho^2 e^{\varrho}.$$

Thus we have for $j=1, 2, \dots, s$

$$\|r_j\| \leqq \frac{1}{2}\left(\frac{m_j}{m}\right)^2 \varrho^2 e^{\varrho} + \frac{m_j}{m}\delta. \tag{6}$$

Next, let $z_j = 1 + \frac{m_j}{m} g$, and consider the difference $T(\mathbf{X}) - z_1 z_2 \dots z_s = y_1 y_2 \dots y_s - z_1 z_2 \dots z_s$. As a polynomial in g and the r_j, it has again non-negative coefficients, so we apply principle (A). The norms of r_j are majorized by (6), and so by the inequality (C)

$$1 + \frac{m_j}{m}\|g\| + \|r_j\| \leqq e^{\frac{m_j}{m}\|g\| + \frac{1}{2}\left(\frac{m_j}{m}\right)^2 \varrho^2 e^{\varrho} + \frac{m_j}{m}\delta},$$

and

$$-\left(1 + \frac{m_j}{m}\|g\|\right) \leqq -e^{\frac{m_j}{m}\|g\| - \frac{1}{2}\left(\frac{m_j}{m}\right)^2 \|g\|^2},$$

where in the last step we used $\frac{m_j}{m}\|g\| \leqq \sqrt{\eta}\,\|g\| \leqq 1$. Thus we see that

$$\|y_1 y_2 \dots y_s - z_1 z_2 \dots z_s\| \leqq e^{\|g\|}\left(e^{\delta + \frac{1}{2}\eta\varrho^2 e^{\varrho}} - e^{-\frac{1}{2}\eta\|g\|^2}\right).$$

Arguing analogously, we have also

$$\|\exp(g) - z_1 z_2 \dots z_s\| \leqq e^{\|g\|}\left(1 - e^{-\frac{1}{2}\eta\|g\|^2}\right).$$

The last two inequalities together imply the conclusion of the lemma.

We note that Theorem 1 is an immediate consequence of the lemma, since the estimate for $\|T(\mathbf{X}_k)-\exp(g)\|$ supplied by the lemma tends to zero as $k\to\infty$ if the hypotheses of the theorem are fulfilled.

3. Proof of Theorem 2

The idea of the proof of Theorem 2 is to find an appropriate sequence of partitions π_k $(k=1, 2, 3, \ldots)$ such that if we let $\delta_k=\delta(\mathbf{X}_k, \pi_k, g)$ then

$$\mathbf{P}\{\lim_{k\to\infty} \delta_k = 0\} = 1, \tag{7}$$

and at the same time such that

$$\lim_{k\to\infty} \eta(\pi_k) = 0. \tag{8}$$

Indeed, if (7) and (8) are fulfilled then the conclusion of Theorem 2 follows from Theorem 1.

Let $C_1 < C_2$ and $\beta < 1$ be three positive constants. We define the partition π_k of $(1, 2, 3, \ldots, k)$ into successive subsequences of lengths $m_j=m_j(k)\,(j=1, 2, \ldots, s=s(k))$ in such a manner that for all j and k

$$C_1 k^\beta < m_j < C_2 k^\beta. \tag{9}$$

Since $\Sigma_j m_j=k$, it follows that

$$s=s(k)=O(k^{1-\beta}), \tag{10}$$

and therefore

$$\eta_k = \eta(\pi_k) = \sum_{j=1}^{s(k)} \left(\frac{m_j}{k}\right)^2 = O(k^{\beta-1}), \tag{11}$$

so that (8) holds. Note that in our probabilistic set-up the partitions π_k are not random variables (i.e. π_k is constant over the whole probability space).

Next, we remark that, by virtue of the Borel—Cantelli lemma, in order to prove (7) it is sufficient to prove that

$$\sum_{k=1}^{\infty} \mathbf{P}\{\delta_k \geqq \varepsilon\} < \infty \tag{12}$$

for any positive ε.

Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_s$ be the successive subsequences of $\mathbf{X}_k$ produced by the partition π_k. Let $N_{j\nu}$ $(j=1, 2, \ldots, s;\ \nu=1, 2, \ldots, \sigma)$ be the number of occurrences of a_ν among the elements of the subsequence $\mathbf{Y}_j$. The $N_{j\nu}$ are random variables subject to the multinomial distribution determined by the probabilities p_ν, and for different j they are independent. We have

$$\delta_k = \mathrm{Max}_{1\leqq j\leqq s} \left\| \sum_{\nu=1}^{\sigma} \left(\frac{N_{j\nu}}{m_j} - p_\nu\right) a_\nu \right\| \leqq AM_k,$$

where

$$A = \mathrm{Max}_\nu \|a_\nu\| \quad \text{and} \quad M_k = \mathrm{Max}_{1 \leqq j \leqq s} \sum_{\nu=1}^{\sigma} \left| \frac{N_{j\nu}}{m_j} - p_\nu \right|.$$

Thus we need to show

$$\sum_{k=1}^{\infty} \mathbf{P}\{M_k \geqq \varepsilon\} < \infty. \tag{13}$$

Since the following inclusion (implication) of events holds

$$\{M_k \geqq \varepsilon\} = \bigcup_{j=1}^{s(k)} \left\{ \sum_{\nu=1}^{\sigma} \left| \frac{N_{j\nu}}{m_j} - p_\nu \right| \geqq \varepsilon \right\} \subseteqq \bigcup_{j=1}^{s(k)} \bigcup_{\nu=1}^{\sigma} \left\{ \left| \frac{N_{j\nu}}{m_j} - p_\nu \right| \geqq \frac{\varepsilon}{\sigma} \right\},$$

we obtain for the probabilities of the complementary events

$$\mathbf{P}\{M_k < \varepsilon\} \geqq \mathbf{P} \bigcap_{j=1}^{s(k)} \bigcap_{\nu=1}^{\sigma} \left\{ \left| \frac{N_{j\nu}}{m_j} - p_\nu \right| < \frac{\varepsilon}{\sigma} \right\} = \tag{14}$$

$$= \prod_{j=1}^{s(k)} \mathbf{P} \bigcap_{\nu=1}^{\sigma} \left\{ \left| \frac{N_{j\nu}}{m_j} - p_\nu \right| < \frac{\varepsilon}{\sigma} \geqq \prod_{j=1}^{s(k)} \left[1 - \sum_{\nu=1}^{\sigma} \mathbf{P} \left\{ \left| \frac{N_{j\nu}}{m_j} - p_j \right| \geqq \frac{\varepsilon}{\sigma} \right\} \right].$$

The equality in (14) is due to the fact that we are dealing with the intersection (conjunction) of independent events.

Suppose N is the number of "successes" in a sequence of m Bernoulli trials with probability p for success. We have then the following fact [3]: given any $\alpha > 1$, for all sufficiently large m (depending only on α and p)

$$\mathbf{P} \left\{ \frac{|N - mp|}{[mp(1-p)]^{1/2}} \geqq (2\alpha \log m)^{1/2} \right\} < \frac{1}{m^\alpha}.$$

It follows that for any $\varepsilon > 0$

$$\mathbf{P} \left\{ \left| \frac{N}{m} - p \right| \geqq \varepsilon \right\} < \frac{1}{m^\alpha} \tag{15}$$

for all sufficiently large m (depending only on ε, α and p). If the inequality (15) is used for the probabilities on the right hand side of (14) we obtain

$$\mathbf{P}\{M_k < \varepsilon\} \geqq \prod_{j=1}^{s(k)} \left[1 - \frac{\sigma}{m_j^\alpha} \right]$$

which is valid for all sufficiently large k, since (9) implies that then all the m_j are large enough. But (9) and (10) show that for suitable constants C and C'

$$\mathbf{P}\{M_k < \varepsilon\} \geqq (1 - Ck^{-\alpha\beta})^{C' k^{1-\beta}} = 1 - O(k^{-\gamma}) \tag{16}$$

where $\gamma = \alpha\beta + \beta - 1$. Since $\alpha > 1$ was arbitrary we may suppose $\gamma > 1$, so that $\mathbf{P}\{M_k \geqq \varepsilon\} = O(k^{-\gamma})$ and (13) follows. This concludes the proof of Theorem 2.

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Hyperinvariant subspaces for n-normal operators

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1. Introduction. Let $\mathfrak{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. (In what follows, all Hilbert spaces will be complex, and all operators under discussion will be bounded and linear.) A closed subspace $\mathfrak{M}\subset\mathfrak{H}$ is said to be *hyperinvariant* for an operator A in $\mathscr{L}(\mathfrak{H})$ if it is non-trivial and is invariant for every operator which commutes with A; that is, if $\mathfrak{M}$ is distinct from $\{0\}$ and $\mathfrak{H}$ and $B(\mathfrak{M})\subset\mathfrak{M}$ for every operator B in $\mathscr{L}(\mathfrak{H})$ satisfying $AB=BA$.

The notion of hyperinvariant subspace was introduced by SZ.-NAGY and FOIAŞ (under the name "ultrainvariant") [10]; these authors and later DOUGLAS and PEARCY [3], [4] characterized the hyperinvariant subspaces of certain types of operators and gave a number of sufficient conditions for an invariant subspace to be hyperinvariant.

The principal purpose of this paper is to show (Theorem 5. 3) that every operator which is *n-normal*, in a sense to be defined below, has a hyperinvariant subspace. Let $\mathfrak{H}^n = \mathfrak{H}\oplus\mathfrak{H}\oplus\cdots\oplus\mathfrak{H}$ denote the orthogonal sum of n copies of the Hilbert space $\mathfrak{H}$. One knows that every operator in $\mathscr{L}(\mathfrak{H}^n)$ can be written as an $n\times n$ matrix $(A_{ij})_{i,j=1}^n$ where each A_{ij} $(1\leqq i,j\leqq n)$ belongs to $\mathscr{L}(\mathfrak{H})$. An operator B on a Hilber space $\mathfrak{K}$ is said to be n-normal if there is a Hilbert space $\mathfrak{H}$ and n^2 mutually commuting normal operators A_{ij} $(1\leqq i,j\leqq n)$ acting on $\mathfrak{H}$ such that $\mathfrak{K}=\mathfrak{H}^n$ and $B=(A_{ij})_{i,j=1}^n$.

The class of n-normal operators may be defined equivalently using the concept of *von Neumann algebras,* i.e. of weakly closed, self-adjoint algebras of operators on Hilbert space, containing the identity operator. If $\mathscr{A}$ is an abelian von Neumann algebra acting on $\mathfrak{H}$ then $M_n(\mathscr{A})$ will denote the von Neumann algebra consisting of all $n\times n$ matrices with entries from $\mathscr{A}$ acting on $\mathfrak{H}^n$ in the usual fashion. It is immediate that an operator A on a Hilbert space is n-normal if and only if there

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exists an abelian von Neumann algebra $\mathscr{A}$ such that A belongs to $M_n(\mathscr{A})$. (It is worth noting that there are operators which are n-normal in the sense of [7] but are not n-normal in our sense. (See § 6.)

The proof that every n-normal operator has a hyperinvariant subspace is accomplished in two steps. First we show that if A and B are quasi-similar operators and if B has a hyperinvariant subspace, then so does A. (See Section 2.) Next, the algebra $M_n(\mathscr{A})$ is identified with an algebra of continuous matrix-valued functions on an extremally disconnected compact Hausdorff space. Using this identification and the techniques developed in [1], [2], and [8], every operator is shown to be quasi-similar to an operator J in $M_n(\mathscr{A})$ which is in "Jordan form". These operators are easily seen to be spectral operators in the sense of Dunford, and thus to have hyperinvariant subspaces [5].

2. Quasi-similarity of operators. If A and B are operators on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively, then A and B are said to be *quasi-similar* if there are bounded linear operators R: $\mathfrak{K}\to\mathfrak{H}$ and S: $\mathfrak{H}\to\mathfrak{K}$ which satisfy the following conditions:

1. $SA=BS$ and $AR=RB$.
2. R and S have zero kernels and dense ranges.

In [11], Sz.-Nagy and Foiaş prove that a quasi-similarity between an operator A and a unitary operator U induces a one to one, order preserving map of the lattice of hyperinvariant subspaces of U into that of A. Examination of their argument also yields the following:

Theorem 2. 1. *If A and B are quasi-similar operators on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively, and if B has a hyperinvariant subspace, then so does A.*

Proof. Let R and S be the operators which invoke the quasi-similarity and let $\mathfrak{M}$ be the subspace of $\mathfrak{K}$ which is hyperinvariant for B. Define:

$$a(\mathfrak{M})=\overline{R(\mathfrak{M})} \quad \text{and} \quad b(\mathfrak{M})=\{f\in\mathfrak{H}\mid S(f)\in\mathfrak{M}\}.$$

Because $\mathfrak{M}\neq\mathfrak{K}$ and S has dense range, $b(\mathfrak{M})\neq\mathfrak{H}$. Also, $a(\mathfrak{M})\neq\{0\}$ since R is one to one and $\mathfrak{M}\neq\{0\}$.

If T is an operator on $\mathfrak{H}$ which commutes with A, then

$$BSTR=SATR=STAR=STRB.$$

Thus STR commutes with B and therefore $STR(\mathfrak{M})\subset\mathfrak{M}$. It follows that $Ta(\mathfrak{M})=\overline{TR(\mathfrak{M})}\subset b(\mathfrak{M})$. Define $q(\mathfrak{M})$ to be the smallest closed subspace of $\mathfrak{H}$ which contains $Ta(\mathfrak{M})$ for every T in $\mathscr{L}(\mathfrak{H})$ that commutes with A. Then $\{0\}\neq a(\mathfrak{M})\subset q(\mathfrak{M})\subset b(\mathfrak{M})\neq\mathfrak{H}$ and $q(\mathfrak{M})$ is hyperinvariant for A.

Examination of the above argument shows that if B has two distinct hyperinvariant subspaces $\mathfrak{M}$ and $\mathfrak{N}$ with $\mathfrak{N} \subset \mathfrak{M}$, then $q(\mathfrak{N}) \subset q(\mathfrak{M})$. However, the following can be proved using the argument of [10, Proposition II. 5. 1]:

Suppose A, B and q are as in Theorem 2. 1 and suppose B is normal. If $\mathfrak{M}$ and $\mathfrak{N}$ are distinct hyperinvariant subspaces for B, then $q(\mathfrak{M}) \neq q(\mathfrak{N})$.

3. Quasi-similarity in $M_n(X)$. In what follows, X will denote an extremally disconnected compact Hausdorff space; i.e., the closure of every open subset of X is compact and open. Such topological spaces are called Stonian spaces, and they arise naturally as the maximal ideal spaces of von Neumann algebras. They may be characterized as those compact Hausdorff spaces whose lattice of real continuous functions is complete [9], and they are known to have a base of compact open sets. Let M_n denote the set of all $n \times n$ complex matrices viewed as the algebra of operators on n-dimensional complex Hilbert space. By $M_n(X)$ we will mean the algebra of all continuous M_n-valued functions $A(\cdot)$ on X with operations defined pointwise and

$$\|A(\cdot)\| = \sup_{x \in X} \|A(x)\|.$$

It is easy to verify that $M_n(X)$ is a C^*-algebra.

Elements of $M_n(X)$ may also be viewed as $n \times n$ matrices with entries in $C(X)$, the algebra of all continuous, complex valued function defined on X. The algebras $M_n(X)$ have been studied extensively by Pearcy and Deckard in [1], [2], and [8].

A matrix $R(\cdot)$ in $M_n(X)$ is said to be *quasi-invertible* if whenever $D(\cdot)$ in $M_n(X)$ satisfies $D(\cdot)R(\cdot)=0$ or $R(\cdot)D(\cdot)=0$ then $D(\cdot)=0$. Matrices $A(\cdot)$ and $B(\cdot)$ are said to be *quasi-similar in* $M_n(X)$ if there exist quasi-invertible matrices $R(\cdot)$ and $S(\cdot)$ in $M_n(X)$ which satisfy

$$S(\cdot)A(\cdot)=B(\cdot)S(\cdot) \quad \text{and} \quad A(\cdot)R(\cdot)=R(\cdot)B(\cdot).$$

Lemma 3. 1. *Suppose that $A(\cdot)$ and $B(\cdot)$ are matrices in $M_n(X)$ and that for each x in a dense subset D of X, $A(x)$ is similar to $B(x)$. Then for each open subset U of X there is a compact open set $V \subset U$ and an invertible element $S(\cdot)$ in $M_n(V)$ such that $S(x)A(x)=B(x)S(x)$ for each x in V.*

Proof. Consider the system L of homogeneous linear equations with coefficients in $C(X)$ which corresponds to the matrix equation $S(\cdot)A(\cdot)=B(\cdot)S(\cdot)$:

$$\text{(L)} \qquad \begin{array}{c} c_{11}s_1 + c_{12}s_2 + \cdots + c_{1m}s_m = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{m1}s_1 + c_{m2}s_2 + \cdots + c_{mm}s_m = 0, \end{array}$$

where $m=n^2$ and the unknown functions s_i represent the entries of $S(\cdot)$ in some prescribed order. For x in X, let $L(x)$ be the corresponding system of scalar equations.

Choose an x_0 in U so that v_0, the rank of $L(x_0)$, is maximal for x in U. There is a $v_0 \times v_0$ minor $N(x_0)$ of the matrix of coefficients of $L(x_0)$ such that $\det N(x_0) \neq 0$. Consequently, $\det N(\cdot)$ does not vanish on a compact open neighborhood V_1 of x_0, $V_1 \subset U$. By the hypothesis, there is an x_1 in $D \cap V_1$ and an invertible complex matrix T_{x_1} satisfying $T_{x_1} A(x_1) = B(x_1) T_{x_1}$. If $u_1, \ldots, u_m$ are the entires in T_{x_1}, then $(u_1 \ldots, u_m)$ is a solution to $L(x_1)$. For i not affiliated with $N(\cdot)$, let $s_i \equiv u_i$ on V_1. For the v_0 values of i affiliated with $N(\cdot)$, use the already assigned s_j and Cramer's rule to define s_i. Since the c_{ij} are continuous and $\det N(x) \neq 0$ for x in V_1, the s_i are continuous and satisfy L. Thus if $S(\cdot)$ is the matrix in $M_n(V_1)$ with the s_i in the appropriate positions, then $S(x)A(x) = B(x)S(x)$ for each x in X.

Since $S(x_1) = T_{x_1}$ is invertible, and the set of invertible matrices is open, there is a compact open set $V \subseteqq V_1$ such that the restriction of $S(\cdot)$ to V is invertible in $M_n(V)$.

Theorem 3.2. *If $A(\cdot)$ and $B(\cdot)$ satisfy the conditions of Lemma* 3. 1, *then $A(\cdot)$ and $B(\cdot)$ are quasi-similar in $M_n(X)$.*

Proof. Let $\mathscr{F}$ denote the collection of all families $\{U_\alpha\}_{\alpha \in I}$ of disjoint compact open subsets of X such that for each α in I there is an $S_\alpha(\cdot)$ in $M_n(U_\alpha)$ which is invertible in $M_n(U_\alpha)$ and satisfies $S_\alpha(x)A(x) = B(x)S_\alpha(x)$ for each x in U_α. Order $\mathscr{F}$ by inclusion and use Zorn's lemma to obtain a maximal family $\{U_\alpha\}_{\alpha \in I}$ in $\mathscr{F}$.

If $Y = \bigcup_{\alpha \in I} U_\alpha$ is not dense in X, then by Lemma 3. 1 there is a compact open set $V \subset X - Y$ and an $S(\cdot)$ in $M_n(V)$ which affects a similarity between the restrictions of $A(\cdot)$ and $B(\cdot)$ to V. This contradicts the maximality of $\{U_\alpha\}_{\alpha \in I}$.

By Lemma 2. 1 of [1], there are matrices $S(\cdot)$ and $R(\cdot)$ in $M_n(X)$ which extend each

$$\frac{1}{\|S_\alpha(\cdot)\|} S_\alpha(\cdot) \quad \text{and} \quad \frac{1}{\|S_\alpha^{-1}(\cdot)\|} S_\alpha^{-1}(\cdot)$$

respectively. These matrices satisfy the equalities

$$S(\cdot)A(\cdot) = B(\cdot)S(\cdot) \quad \text{and} \quad A(\cdot)R(\cdot) = R(\cdot)B(\cdot).$$

It remains to show that $R(\cdot)$ and $S(\cdot)$ are quasi-invertible. Suppose $C(\cdot)$ is a matrix in $M_n(X)$ and $C(\cdot)R(\cdot) = 0$; that is, $C(x)R(x) = 0$ for each x in X. For each x in the dense subset Y of X, $R(x)$ is invertible and so $C(x) = 0$. It follows that $C(\cdot) = 0$. The other three implications are easily proved in the same way.

4. Jordan forms in $M_n(X)$. In [2], Deckard and Pearcy exhibit a Stonian space X and a matrix $A(\cdot)$ in $M_2(X)$ for which there is no $J(\cdot)$ in $M_2(X)$ which is similar to $A(\cdot)$ and is such that $J(x)$ is in Jordan form for each x in X. If the condition of similarity is relaxed to quasisimilarity, then such Jordan forms always exist. This is shown via the following lemmas.

Lemma 4.1. *If $\varphi_1, \ldots, \varphi_n$ are in $C(X)$, where X is a Stonian space, and if U is a non-empty, open subset of X, then there is a non-empty, compact, open set $V \subset U$ such that for each i and j ($1 \leqq i,\ j \leqq n$) either $\varphi_i(x) = \varphi_j(x)$ for all x in V, or $\varphi_i(x) \neq \varphi_j(x)$ for all x in V.*

Proof. Pick x_0 in U so that the number of distinct values $\varphi_i(x_0)$ is maximal; assume these values are $\varphi_{i_1}(x_0), \ldots, \varphi_{i_l}(x_0)$. There is an open neighborhood U_0 of x_0, $U_0 \subset U$, such that $\varphi_{i_j}(x) \neq \varphi_{i_k}(x)$ for $j \neq k$ and x in U_0. For each i, $\varphi_i(x_0) = \varphi_{i_j}(x_0)$ for some j and hence $\varphi_i(x_0) \neq \varphi_{i_k}(x_0)$ for $k \neq j$. Therefore $\varphi_i(x) \neq \varphi_{i_k}(x)$ for each k, $k \neq j$ and for each x in some compact open neighborhood V_i of x_0, $V_i \subset U_0$. Consequently, $\varphi_i(x) = \varphi_{i_j}(x)$ for each x in V_i and $V = \bigcap_{i=1}^{n} V_i$ is the desired set.

Lemma 4.2. *If U is non-empty subset of the Stonian space X, and if $B(\cdot)$ is in $M_n(X)$, then there is a non-empty compact open set $V \subset U$ on which the rank of $B(\cdot)$ is constant.*

Proof. Choose an x_0 in U so that r_0, the rank of $B(x_0)$, is maximal for x in U. There is an $r_0 \times r_0$ minor $M(x_0)$ of $B(x_0)$ with $\det M(x_0) \neq 0$, and hence $\det M(\cdot)$ does not vanish in some compact open neighborhood V of x_0, $V \subset U$. It follows that the rank of $B(\cdot) \equiv r_0$ on V.

Suppose that A and A' are $n \times n$ scalar matrices in Jordan form, having the single eigenvalues λ and λ' respectively. More explicitly, suppose $A = \sum_{i=1}^{k} \oplus A_i$ where A_i is an $s_i \times s_i$ Jordan block matrix for λ and $s_{i-1} \leqq s_i$ for $1 < i \leqq k$. Similarly, $A' = \sum_{i=1}^{l} \oplus A_i'$ where A_i' is an $s_i' \times s_i'$ Jordan block for λ' and the A_i' are arranged in order of decreasing size. (A Jordan block for λ is a square matrix with each entry on the main diagonal equal to λ, with ones on the diagonal above the main diagonal, and with zeros in all other positions. A finite scalar matrix is in Jordan form if it is the direct sum of Jordan blocks.)

Lemma 4.3. *If, in the notation established above,* $\operatorname{Rank}(A-\lambda)^r = \operatorname{Rank}(A'-\lambda')^r$ *for each $r \leqq \max\{s_1, s_1'\}$, then $k = l$ and $s_i = s_i'$ for each i, $1 \leqq i \leqq k$.*

Proof. This lemma is proved by induction on $\max\{s_1, s_1'\}$. Since

$$0 = \operatorname{Rank}(A-\lambda)^{s_1} = \operatorname{Rank}(A'-\lambda')^{s_1}$$

and $A'-\lambda'$ is nilpotent of index s_1', $s_1 \geqq s_1'$. Similarly, $s_1' \geqq s_1$ and so $s_1 = s_1'$. Beacuse $\operatorname{Rank}(A-\lambda)^{s_1-1} = \operatorname{Rank}(A'-\lambda')^{s_1-1}$, the number of s_i equal to s_1 is the same as the number of s_j' equal to s_1. Consequently, if m is this number then the matrices

$$B = \operatorname{diag}(A_{m+1}, \ldots, A_k) \quad \text{and} \quad B' = \operatorname{diag}(A'_{m+1}, \ldots, A'_l)$$

satisfy the hypothesis of the lemma and max $\{s_{m+1}, s'_{m+1}\} < s_1$. Therefore, by the induction hypothesis, $k-m = l-m$ and $s_i = s'_i$ for $m+1 \leqq i \leqq k$.

In [1], DECKARD and PEARCY prove that if X is a Stonian space and if $\varrho(\lambda, x) = \lambda^n + a_{n-1}(x)\lambda^{n-1} + \cdots + a_2(x)\lambda + a_0(x)$ is a monic polynomial with coefficients in $C(X)$, then there is a function φ in $C(X)$ such that $\varrho(\varphi(x), x) = 0$ for every x in X. It follows that all such polynomials can be written in the form $\varrho(\lambda, x) = \prod_{i=n}^{n} (\lambda - \varphi_i(x))$ where the functions φ_i $(1 \leqq i \leqq n)$ are in $C(X)$. Of particular interest in this paper is the case when $\varrho(\lambda, x)$ is the characteristic polynomial of some matrix $A(\cdot)$ in $M_n(X)$:

$$\varrho(\lambda, x) = \det [\lambda I - A(x)].$$

In this case, $\varrho(A(x), x) = 0$ for each x in X.

Lemma 4.4. *If U is a non-empty open subset of the Stonian space X, and if $A(\cdot)$ is a matrix in $M_n(X)$, then there is a non-empty, compact, open set V contained in U and a matrix $J(\cdot)$ in $M_n(V)$ such that $J(x)$ is in Jordan form and is similar to $A(x)$ for each x in V.*

Proof. By virtue of Lemmas 4.1 and 4.2 there is a compact open set $V \subset U$ which satisfies the following conditions:

1. The restriction to V of the characteristic polynomial $\varrho(\lambda, \cdot)$ of $A(\cdot)$ can be factored as

$$\varrho(\lambda, \cdot) = \prod_{i=j}^{k} (\lambda - \varphi_i)^{r_i}$$

where for $i \neq j$, $\varphi_i(x) \neq \varphi_j(x)$ for each x in V.

2. For every set of positive integers

$$\{s_i : 1 \leqq i \leqq k,\ 0 \leqq s_i \leqq r\},$$

the matrix $\prod_{i=1}^{k} (A(\cdot) - \varphi_i)^{s_i}$ has constant rank on V.

For x in X, let $J(x) = \operatorname{diag}(J_x^1, J_x^2, \ldots, J_x^k)$ be the matrix similar to $A(x)$ where J_x^i is a $t_x^i \times t_x^i$ matrix in Jordan form with a single eigenvalue $\varphi_i(x)$ and the Jordan blocks of J_x^i are arranged in order of decreasing size. If x and y are in V, then for $1 \leqq i \leqq k$, J_x^i and J_y^i satisfy the conditions of Lemma 4.3. In fact,

$$t_x^i = n - \operatorname{Rank}(J(x) - \varphi_i(x))^{r_i} = n - \operatorname{Rank}(A(x) - \varphi_i(x))^{r_i} =$$
$$= n - \operatorname{Rank}(A(y) - \varphi_i(y))^{r_i} = n - \operatorname{Rank}(J(x) - \varphi_i(y)) = t_y^i,$$

and, for $s_i \leqq t_x^i$,

$$\operatorname{Rank}(J_x^i - \varphi_i(x))^{s_i} = \operatorname{Rank}(J(x) - \varphi_i(x))^{s_i} - (n - t_x^i) =$$
$$= \operatorname{Rank}(J(x) - \varphi_i(x))^{s_i} - (n - t_y^i) = \operatorname{Rank}(J_y^i - \varphi_i(y))^{s_i}.$$

Consequently, J_x^i and J_y^i differ only along the main diagonal, and hence the same is true of $J(x)$ and $J(y)$.

It is now easy to see that the matrix valued function $J(\cdot)$ is continuous; in fact, the only entries along its main diagonal are the functions φ_i, and the entries in all other positions are constant functions. Thus $J(\cdot)$ is in $M_n(V)$, and the proof is complete.

Theorem 4. 5. *If X is a Stonian space and $A(\cdot)$ is a matrix in $M_n(X)$, then there is a $J(\cdot)$ in $M_n(X)$ such that $J(x)$ is in Jordan form for each x in X and such that for each x in a dense subset D of X, $J(x)$ is similar to $A(x)$.*

Proof. Let $\mathscr{F}$ be the family of all collections $\{U_\alpha\}_{\alpha\in I}$ of non-empty disjoint, compact open subsets of X where for each α in I there is a $J_\alpha(\cdot)$ in $M_n(U_\alpha)$ such that $J_\alpha(x)$ is in Jordan form and similar to $A(x)$ for each x in U_α. Order $\mathscr{F}$ by inclusion and use Zorn's lemma to obtain a maximal family $\{U_\alpha\}_{\alpha\in I}$ in $\mathscr{F}$. If $D = \bigcup\limits_{\alpha\in I} U_\alpha$ is not dense in X, then by Lemma 4. 4, there is a non-empty compact open set V contained in $X-\bar{D}$ and a $J(\cdot)$ in $M_n(V)$ which pointwise is in Jordan form and is similar to $A(\cdot)$. But this contradicts the maximality of $\{U_\alpha\}_{\alpha\in I}$. Therefore D is dense in X. For each α in I and x in U_α, the entries in $J_\alpha(x)$ are all bounded by $\max\{\|\varphi\|, \ldots, \|\varphi_n\|, 1\}$; therefore, by Lemma 2. 1 of [1], there is a $J(\cdot)$ in $M_n(X)$ which extends each $J_\alpha(\cdot)$.

It remains to show that for x_0 in $X-D$, $J(x_0)$ is in Jordan form. Suppose $J(\cdot)=(J_{ij})_{i,j=1}^n$ where each J_{ij} is in $C(X)$; then for $j\neq i, i+1$, J_{ij} is the zero function and for $j=i+1$, $J_{ij}(x_0)$ is either zero or one. Suppose $J_{i,i+1}(x_0)=1$, then if $\langle x_\beta\rangle$ is any net in D which converges to x_0, $J_{i,i+1}(x_\beta)$ converges to 1. Since for each β, $J_{i,i+1}(x_\beta)$ is either zero or one, the net $\langle J_{i,i+1}(x)_\beta\rangle_\beta$ is eventually identically equal to one. But each $J(x_\beta)$ is in Jordan form, so eventually, $J_{ii}(x_\beta)=J_{i+1,i+1}(x_\beta)$ and hence $J_{i,i}(x_0)=J_{i+1,i+1}(x_0)$. Therefore $J(x_0)$ is in Jordan form.

Using Theorems 3. 2 and 4. 5, the following is obtained:

Theorem 4. 6. *If X is a Stonian space and $A(\cdot)$ is a matrix in $M_n(X)$, then there is a $J(\cdot)$ in $M_n(X)$ such that $J(x)$ is in Jordan form for each x, and $J(\cdot)$ is quasi-similar to $A(\cdot)$ in $M_n(X)$.*

5. An application to $M_n(\mathscr{A})$. If $\mathscr{A}$ is an abelian von Neumann algebra acting on the Hilbert space $\mathfrak{H}$, then the maximal ideal space X of $\mathscr{A}$ is Stonian and the Gelfand map $\Gamma:\mathscr{A}\to C(X)$ is a *-isometrical isomorphism. Let $M_n(\mathscr{A})$ denote the von Neumann algebra consiting of all $n\times n$ matrices with entries in $\mathscr{A}$ acting on $\mathfrak{H}^n$ in the usual fashion. To each $A=(A_{ij})_{i,j=1}^n$ in $M_n(\mathscr{A})$ there corresponds a natural element $A(\cdot)$ in $M_n(X)$,

$$A(\cdot) = (\Gamma(A_{ij}))_{i,j=1}^n.$$

This correspondence is clearly a *-isomorphism.

If R is an operator in $M_n(\mathscr{A})$, then the kernel of R is the kernel of R^*R and the projection P onto the kernel of R is a spectral projection for R^*R and thus lies in the von Neumann algebra $M_n(\mathscr{A})$. Therefore if the kernel of R is larger than $\{0\}$, there is a non-zero operator P in $M_n(\mathscr{A})$ satisfying $RP=0$. By taking adjoints, one sees that if the range of R is not dense, there is a non-zero P in $M_n(\mathscr{A})$ such that $PR=0$. It follows that an operator R in $M_n(\mathscr{A})$ has has zero kernel and dense range if and only if the corresponding element $R(\cdot)$ of $M_n(X)$ is quasi-invertible in $M_n(X)$. Therefore, if $A(\cdot)$ and $B(\cdot)$ are quasi-similar matrices in $M_n(X)$, then A and B are quasi-similar as operators on $\mathfrak{H}^n$. This observation yields the following theorem.

Theorem 5. 1. *If $\mathscr{A}$ is an abelian von Neumann algebra and if A is an operator in $M_n(\mathscr{A})$, then there is an operator J in $M_n(\mathscr{A})$ which is in Jordan form and is quasi-similar to A. That is,*

$$J = \begin{pmatrix} J_1 & P_1 & & & \\ & J_2 & P_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & P_{n-1} \\ & & & & J_n \end{pmatrix},$$

where $J_i\ (1 \leqq i \leqq n)$ and $P_j\ (1 \leqq j \leqq n-1)$ are in $\mathscr{A}$ and the operators P_j are projections.

In [6], S. R. Foguel obtains a similar result using measure theoretic techniques.

A matrix of complex numbers which is in Jordan form can be written in an obvious way as the sum of a diagonal matrix and a nilpotent matrix. A simple calculation shows that the diagonal part and the nilpotent part commute. This observation has its obvious analog for matrices in $M_n(X)$. Using this analog, and the relationship between $M_n(\mathscr{A})$ and $M_n(X)$, the following is obtained:

Corollary 5. 2. *Every operator A in $M_n(\mathscr{A})$ is quasi-similar to an operator $D+N$ where D and N are commuting operators in $M_n(\mathscr{A})$, D is normal, and N is nilpotent.*

We are now in a position to prove the basic theorem of this paper.

Theorem 5. 3. *Every non-scalar n-normal operator A on a Hilbert space $\mathfrak{H}$ has a hyperinvariant subspace.*

Proof. By virtue of Theorem 2. 1 and Corollary 5. 2, we may assume that $A = D+N$, where D is a normal operator, N is a nilpotent operator, and D and N commute. In [5], Dunford shows that such operators are spectral operators, and hence if T is an operator which commutes with A, then T commutes with the resolution of the identity for A; that is, T commutes with the spectral projections for D. Therefore, if D is not a multiple of the identity operator, D has spectral projections distinct from 0 and I, and the ranges of the projections are hyperinvariant subspaces for A.

In case D is scalar, then a subspace $\mathfrak{M}$ of $\mathfrak{H}$ will be hyperinvariant for A just in case it is hyperinvariant for N. But A is non-scalar, so N cannot be the zero operator. On the other hand, N is nilpotent so $N(\mathfrak{H})$ is not dense in $\mathfrak{H}$. Therefore $\mathfrak{M}=\overline{N(\mathfrak{H})}$ is a hyperinvariant subspace for N and hence for A.

A simple argument extends Theorem 5. 3 to direct sums of n-normal operators. If α is a non-zero scalar, then the scalar matrices

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & \alpha & & & \\ & \lambda & \alpha & & \\ & & \ddots & \ddots & \\ & & & \ddots & \alpha \\ & & & & \lambda \end{pmatrix}$$

are similar. In fact, the former is the Jordan form for the latter. Thus if J is a scalar matrix in Jordan form, and J is written as $D+N$ where D is a diagonal matrix and N has zero entries in every position except perhaps on the diagonal above the main diagonal where N may have some ones, then J is similar to $D+\alpha N$. Now if X is a Stonian space, and $J(\cdot)$ as an element of $M_n(X)$ is pointwise in Jordan form, then by writing $J(\cdot) = D(\cdot)+N(\cdot)$ as in the scalar case and by using Theorem 3. 2, we get that $J(\cdot)$ is quasi-similar in $M_n(X)$ to $D(\cdot)+\alpha N(\cdot)$. Thus, using the representation of $M_n(\mathscr{A})$ as $M_n(X)$, we see that in Corollary 5. 2 if N is not zero then we can arrange things so that the norm of N is any positive number.

Theorem 5. 4. *If for each integer* $i\geqq 0$, A_i *is a possibly zero i-normal operator acting on the Hilbert space* $\mathfrak{H}_i$, *and if* A *is the direct sum operator* $\Sigma\oplus A_i$, *then if* A *is non-scalar* A *has a hyperinvariant subspace.*

Proof. Each A_i is quasi-similar to an operator D_i+N_i where D_i and N_i commute, D_i is normal, N_i is nilpotent of index at most i and $\|N_i\|\leqq 1/i$. Thus A is quasi-similiar to $D+N$ where $D = \Sigma\oplus D_i$ and $N = \Sigma\oplus N_i$. Clearly, D is normal and commutes with N. Furthermore,

$$\|N^n\| = \sup_i \|N_i^n\| = \sup_{i>n} (1/i)^n \leqq (1/n)^n.$$

Therefore $\|N^n\|^{1/n}$ converges to zero, N is quasinilpotent, and $D+N$ is a spectral operator. If D is not a multiple of the identity operator, then the spectral subspaces for D are hyperinvariant for $D+N$. If D is a scalar operator, then N, as a non-zero direct sum of nilpotent operators, will have many hyperinvariant subspaces and these will be hyperinvariant for $D+N$ also. In any case, $D+N$ has hyperinvariant subspaces, and, by Theorem 2. 1 so does A.

Translated into the language of von Neumann algebras, Theorem 5. 4 says that every operator which belongs to a type I finite von Neumann algebra has a hyperinvariant subspace.

6. The term n-normal has been used with a somewhat broader meaning than that which we have given it. The algebras $M_n(\mathscr{A})$, where $\mathscr{A}$ is an abelian von Neumann algebra, are commonly said to be of type I_n, and a von Neumann algebra is n-normal if it is the direct sum of algebras of type I_k where $k \leqq n$. According to [8] an operator is n-normal if the von Neumann algebra it generates is n-normal. To avoid confusion, we will say that operators which are n-normal in this latter sense are of *type n.*

In [7], for example, a von Neumann algebra $\mathscr{V}$ is equivalently defined as n-normal if it is satisfies the identity $\Sigma \pm X_{i_1} X_{i_2} \dots X_{i_{2n}} = O$, where X_i $(i=1, \dots, 2n)$ are arbitrary elements of $\mathscr{V}$, the sum is taken over all permutations of $(1, 2, \dots, 2n)$, and the sign is determined by the parity of the permutation. This characterization makes it clear that any von Neumann subalgebra of an n-normal algebra is n-normal, and thus that an operator which is n-normal in our sense is of type n. That the converse is false can be seen in the following example.

Let $\mathscr{A}$ denote the multiplication algebra acting on L^2 of the unit circle with Lebesgue measure, and let $\mathscr{C}$ be the algebra of all operators on one-dimensional Hilbert space. If $\mathscr{V}$ denotes the direct sum algebra $\mathscr{C} + M_2(\mathscr{A})$ and if T is the operator

$$I \oplus \begin{pmatrix} -1 & S \\ O & O \end{pmatrix}$$

where S is multiplication by the coordinate function, then T generates $\mathscr{V}$, and thus T is of type 2. But T is not n-normal for any n; for suppose it were. Then T and hence $\mathscr{V}$ are contained in some $M_n(\mathscr{W})$ where $\mathscr{W}$ is an abelian von Neumann algebra, and, since T is not normal, n is at least 2. Next consider the rank one projection P which is the direct sum of the identity element of C and the zero element of $M_2(\mathscr{A})$. By Theorem 1 of [8], P is unitarily equivalent to a diagonal element D of $M_n(\mathscr{W})$. If $D_1, \dots, D_n$ are the diagonal entries in D, then for some j, $1 \leqq j \leqq n$, D_j is a rank one projection in $\mathscr{W}$, and the diagonal operator E, all of whose main diagonal entries are equal to D_j, is a rank n projection which commutes with $M_n(\mathscr{W})$ and hence with $\mathscr{V}$. On the other hand, the commutant of $\mathscr{V}$ consists of all operators of the form

$$\lambda I \oplus \begin{pmatrix} A & O \\ O & A \end{pmatrix}$$

where λ is any scalar and A is in $\mathscr{A}$. In the particular, if the commuting operator is a projection, then λ is 0 or 1 and A is multiplication by the characteristic function of some measurable set. The only way such a projection can be of finite rank is for A to be the zero operator, and so the only non-zero, finite rank projection which commutes with the algebra $\mathscr{V}$ is the rank one projection P. But E is a projection of rank $n\neq 1$ and E commutes with $\mathscr{V}$. This is a contradiction and thus the original assumption, that T was n-normal, is false.

Fortunately, this confusion over definitions causes no problems with our Theorem 5. 3, for if T is a non-scalar operator of type n, then T is a direct sum of operators which are i-normal for some i and hence, by Theorem 5. 4, T has a hyperinvariant subspace.

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Hyperinvariant subspaces for spectral and n-normal operators

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If A is a bounded linear operator on a complex Banach space, then a closed linear subspace $\mathfrak{M}$ is *hyperinvariant* for A if $\mathfrak{M}$ is invariant under every operator which commutes with A. It is not known whether or not every operator other than a multiple of the identity has a non-trivial hyperinvariant subspace (i.e. other than the zero subspace and the whole space). Several sufficient conditions for the existence of non-trivial hyperinvariant subspaces are known ([3], [13], [14]).

Fuglede's theorem [6] states that every spectral subspace of a normal operator is hyperinvariant, and this was generalized to spectral operators by DUNFORD [4]. HOOVER [5] recently showed that every n-normal operator has a hyperinvariant subspace. In this note we present simple proofs of DUNFORD's and HOOVER's results, based upon Rosenblum's theorem on operator equations.

1. Rosenblum's theorem

We shall use a theorem about solutions of certain linear operator equations. The theorem was proved by ROSENBLUM [11] to the case where E and F are elements of the same Banach algebra. The result which is given below has not, to our knowledge, appeared in print before, although many people must be aware of it. Our proof is essentially the same as the proof of Rosenblum's result contained in the paper of LUMER and ROSENBLUM [9]. We denote the set of bounded inear operators from $\mathscr{X}$ to $\mathscr{Y}$ by $\mathscr{B}(\mathscr{X}, \mathscr{Y})$.

Theorem (Rosenblum). *If E and F are bounded operators on the complex Banach spaces $\mathscr{Y}$ and $\mathscr{X}$ respectively, and if the operator $\mathscr{T}$ on $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ is defined by $\mathscr{T}(X) = EX - XF$, then*

$$\sigma(\mathscr{T}) \subset \sigma(E) - \sigma(F) = \{z - w \colon z \in \sigma(E),\ w \in \sigma(F)\}.$$

Proof (similar to [9]). Define operators $\mathscr{E}$ and $\mathscr{F}$ on $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ by

$$\mathscr{E}(X) = EX \quad \text{and} \quad \mathscr{F}(X) = XF.$$

If $E-\lambda$ has in inverse, then $(E-\lambda)(E-\lambda)^{-1}X = (E-\lambda)^{-1}(E-\lambda)X = X$ for every $X \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$, and therefore $\sigma(\mathscr{E}) \subset \sigma(E)$. Similarly $\sigma(\mathscr{F}) \subset \sigma(F)$.

Since $\mathscr{E}$ and $\mathscr{F}$ are commuting operators, $\sigma(\mathscr{E}-\mathscr{F}) \subset \sigma(\mathscr{E})-\sigma(\mathscr{F})$; (simply let $\mathscr{A}$ be a maximal commutative algebra containing $\mathscr{E}$ and $\mathscr{F}$ and use the fact that the spectra relative to $\mathscr{A}$, which are the ranges of the Gelfand transforms, are the same as the original spectra). Hence $\sigma(\mathscr{T}) \subset \sigma(E)-\sigma(F)$.

The special case of this result that we shall need is the fact that $\sigma(E) \cap \sigma(F) = \emptyset$ and $EX_0 = X_0F$ imply $X_0 = 0$ (since X_0 is in the nullspace of the operator $\mathscr{T}(X) = EX-XF$).

2. The Fuglede—Dunford theorem

Fuglede's theorem [6] states that whenever a bounded operator B on a Hilbert space commutes with a normal operator A, then B commutes with the spectral measure of A (or, equivalently, then B commutes with A^*). Halmos [7, 8] and Rosenblum [12] gave simplified proofs of Fuglede's theorem. Dunford [4] generalized Fuglede's theorem to the case where A is a spectral operator on a Banach space.

In this note we give another proof of Dunford's version of the theorem. We feel that this proof gives some further insight even in the Hilbert space case, although it is neither as short nor as elegant as Rosenblum's proof.

Following Dunford [4] we say that a bounded operator A on a Banach space $\mathscr{X}$ is a *spectral operator* if there exists a spectral measure $E(\cdot)$ (i.e. a countably additive mapping from the Borel sets in the complex plane into a uniformly bounded family of projections on $\mathscr{X}$ such that $E(\emptyset)=0$, $E(C)=1$, and $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$ for all Borel sets σ_1 and σ_2), which commutes with A and which has the property that $\sigma(A|E(\sigma)\mathscr{X}) \subset \bar{\sigma}$ for all Borel sets σ.

Theorem (Fuglede—Dunford). *If A is a spectral operator with spectral measure $E(\cdot)$, and if $AB=BA$, then $BE(\sigma)=E(\sigma)B$ for all Borel sets σ.*

Proof. It obviously suffices to show that the range of $E(\sigma)$ is invariant under B for each Borel set σ, and this is equivalent to showing that $E(\sigma')BE(\sigma)=0$, where σ' denotes the complement of σ. By regularity it suffices to show that $E(\sigma')BE(\sigma)=0$ whenever σ is closed.

Fix a closed set σ, and let σ_0 be any closed subset of σ'. From $AB=BA$ it follows that $E(\sigma_0)AB\,E(\sigma) = E(\sigma_0)B\,AE(\sigma)$ and thus that

$$[E(\sigma_0)A\,E(\sigma_0)]\,[E(\sigma_0)B\,E(\sigma)] = [E(\sigma_0)\,B\,E(\sigma)]\,[E(\sigma)A\,E(\sigma)].$$

Hence $E(\sigma_0)B\,E(\sigma)=0$ by Rosenblum's theorem, since $E(\sigma_0)A\,E(\sigma_0)$ and

$E(\sigma)A\,E(\sigma)$ have disjoint spectra as operators on $E(\sigma_0)\mathscr{X}$ and $E(\sigma)\mathscr{X}$ respectively.

Since $E(\sigma_0)B\,E(\sigma)=0$ whenever σ_0 is a closed subset of σ', it follows that $E(\sigma')B\,E(\sigma)=0$ and the proof is complete.

3. Putnam's corollary

Soon after Fuglede's theorem was published, PUTNAM [10] observed that Fuglede's proof could be generalized to show that whenever A and C are normal operators on a Hilbert space and B is a bounded operator such that $AB=BC$ then $A^*B=BC^*$. BERBERIAN [1] found a simple trick for getting Putnam's result as a corollary of Fuglede's. To our knowledge it has not previously been observed that Berberian's trick can be applied to the case of spectral operators, yielding the following result.

Corollary. *If A and C are spectral operators with spectral measures $E(\cdot)$ and $F(\cdot)$ on the Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ respectively, and if B is a bounded operator from $\mathscr{Y}$ to $\mathscr{X}$ such that $AB=BC$, then $E(\sigma)B=BF(\sigma)$ for every Borel set σ.*

Proof. We consider $\mathscr{X}\oplus\mathscr{Y}$, with $\|(x,y)\| = \|x\|+\|y\|$, and let P and Q be the projections onto the first and second co-ordinate spaces respectively. Then the operator $T = PAP+QCQ$ is spectral, and its spectral measure is defined by $G(\sigma) = PE(\sigma)P+QF(\sigma)Q$ for each σ. A trivial computation shows that T commutes with the operator $S=PBQ$. By the Fuglede—Dunford Theorem, $G(\sigma)S=SG(\sigma)$ for each σ. Another simple computation gives $B\,E(\sigma)=F(\sigma)B$.

4. Hyperinvariant subspaces of triangular and n-normal operators

An operator is said to be *n-normal* if it is (unitarily equivalent to) an operator in the tensor product of some abelian von Neumann algebra and the algebra of $n\times n$ matrices. In other words, n-normal operators can be written in the form

$$\begin{pmatrix} A_{11} & A_{12}\dots A_{1n} \\ \vdots & \\ A_{n1} & A_{n2}\dots A_{nn} \end{pmatrix}$$

where $\{A_{ij}\}$ is a collection of commuting normal operators.

R. G. DOUGLAS and C. PEARCY showed that every 2-normal operator has a non-trivial hyperinvariant subspace, and T. B. HOOVER [5] generalized this result to n-normal operators. Hoover shows that every n-normal operator is quasi-similar to an n-normal operator in "Jordan form", and derives the existence of hyperinvariant subspaces from this result together with Dunford's characterization of spectral operators.

We show that the existence of hyperinvariant subspaces for n-normal operators follows from the more easily proven result that every n-normal operator is unitarily equivalent to an n-normal operator in upper triangular form [2], together with the simple theorem given below.

Theorem. *If A is unitarily equivalent to an operator in the upper triangular form*

$$(*)\qquad \begin{pmatrix} A_{11} & * \dots * \\ 0 & * \dots * \\ \vdots & \\ 0 & 0 \dots A_{nn} \end{pmatrix},$$

where the spectra of A_{11} and A_{nn} are disjoint, then A has a non-trivial hyperinvariant subspace.

Proof. Let

$$B = \begin{pmatrix} * & * \dots * \\ \vdots & \vdots \\ B_{n1} & * \dots * \end{pmatrix}$$

be any operator in the commutant $\mathscr{A}$ of A. The fact that the entry in position $(n, 1)$ of AB is equal to the entry in position $(n, 1)$ of BA gives $A_{nn}B_{n1} = B_{n1}A_{11}$. Since the spectra of A_{11} and A_{nn} are disjoint, Rosenblum's theorem implies that $B_{n1} = 0$. Let x be any vector of the form $(x_1, 0, 0, 0, \dots, 0)$ with $x_1 \neq 0$ and y any vector of the form $(0, 0, \dots, 0, y_n)$ with $y_n \neq 0$. We have shown that $(Bx, y) = 0$ for all $B \in \mathscr{A}$. Thus the closure of $\{Bx \colon B \in \mathscr{A}\}$ is a non-trivial hyperinvariant subspace for A.

Corollary. *If A is not a multiple of the identity and is unitarily equivalent to an operator in the upper triangular form $(*)$, where A_{11} and A_{nn} are normal, then A has a non-trivial hyperinvariant subspace.*

Proof. If the spectrum of A_{11} consists of only one point, then A_{11} is a multiple of the identity. In this case A has a non-trivial eigenspace, and it is trivial to verify the fact that an eigenspace of A is hyperinvariant.

If the spectrum of A_{11} consists of more than one point, then, by the spectral theorem, we can write $A_{11} = A_{11}^0 \oplus A_{11}^1$ and $A_{nn} = A_{nn}^0 \oplus A_{nn}^1$ where the spectra of

A_{11}^0 and A_{nn}^1 are disjoint. Then A is unitarily equivalent to an operator of the form

$$\begin{pmatrix} A_{11}^0 & 0 & * & \dots & * \\ 0 & A_{11}^1 & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & & & & \\ 0 & 0 & \dots & A_{nn}^0 & * \\ 0 & & \dots & 0 & A_{nn}^1 \end{pmatrix}.$$

Thus the Theorem above gives the result.

Corollary (Hoover). *Every n-normal operator which is not a multiple of the identity has a non-trivial hyperinvariant subspace.*

Proof. A theorem of Deckard and Pearcy [2, Theorem 2] implies that every n-normal operator is unitarily equivalent to an n-normal operator in upper triangular form. Thus the result follows from the previous corollary.

Remark. As Hoover [5] shows, quasi-similarity preserves the existence of hyperinvariant subspaces. Thus the theorem and the first corollary above can be stated with "unitarily equivalent" replaced by "quasi-similar".

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Operators with bounded characteristic function and their J-unitary dilation

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Introduction. Let $\mathfrak{H}$ be a (complex) Hilbert space and let T be a bounded linear operator on $\mathfrak{H}$.

Denote by Q_T the positive square root of $|I-T^*T|$ and by J_T the operator $\operatorname{sgn}(I-T^*T)$; similarly, let $Q_{T^*}=|I-TT^*|^{\frac{1}{2}}$, $J_{T^*}=\operatorname{sgn}(I-TT^*)$. Let us put

(0.1) $$\Theta_T(\lambda)=[-TJ_T+\lambda Q_{T^*}(I-\lambda T^*)^{-1}Q_T]|\overline{Q_T\mathfrak{H}}$$

whenever $(I-\lambda T^*)^{-1}$ exists. This function, whose values are operators from $\mathfrak{D}_T=\overline{Q_T\mathfrak{H}}$ to $\mathfrak{D}_{T^*}=\overline{Q_{T^*}\mathfrak{H}}$, is called the "characteristic function" of T (see [13], [10]; for the case where T is a contraction, see [15]). The main result of the present paper is the following

Theorem. *If $\Theta_T(\lambda)$ is defined for all λ with $|\lambda|<1$, and if*

$$\sup\{\|\Theta_T(\lambda)\|:|\lambda|<1\}<\infty,$$

then T is similar to a contraction.

Here similarity has the usual meaning: Two operators T, T_1 are called "similar" if there exists an affinity X (i.e. an operator mapping the space of T_1 onto the space of T in a one-to-one and continuous way) such that $T=XT_1X^{-1}$, see [15].

It is of interest to have a boundedness condition which implies similarity of T to a contraction, in view of the fact that the apparently more natural conditions

$$\sup_{n\geq 0}\|T^n\|<\infty,\quad \sup_{|\lambda|>1}(|\lambda|-1)\|(\lambda I-T)^{-1}\|<\infty,$$

formely conjectured to be sufficient for similarity to a contraction, have turned out not to be [4], [7], [8, p. 200].

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However, it is worth while to mention that the existence and boundedness of $\Theta_T(\lambda)$ on $\{\lambda: |\lambda|<1\}$ is not necessary for T being similar to a contraction. Indeed, taking $\mathfrak{H}$ the two dimensional complex Euclidean space E^2 and T the operator corresponding to the matrix $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ one obtains by simple computations that T is similar to the contraction $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, while $Q_T=Q_{T^*}=I$ and $\Theta_T(\lambda)$ is given by the matrix

$$\begin{pmatrix}(1-\lambda)^{-1} & \lambda^2(1-\lambda)^{-1}\\ 1 & \lambda\end{pmatrix}$$

which is unbounded on $\{\lambda: |\lambda|<1\}$.

The theorem is an outgrowth of two known results. The first [15, IX. 1], [6] gives the condition for a contraction to be similar to a unitary. It was generalized by L. A. Sahnovič [12] to apply to general bounded T: *If $\Theta_T(\lambda)$ is defined and bounded on $\{\lambda: |\lambda|\neq 1\}$ then T is similar to a unitary operator.* Our theorem also contains the following similarity theorem of G. C. Rota [11]: *If the spectrum $\sigma(T)$ of T is contained in $\{\lambda: |\lambda|<1\}$, then T is similar to a contraction.* Indeed $\sigma(T)\subset\{\lambda: |\lambda|<1\}$ implies that $\|(I-\lambda T^*)^{-1}\|$ is bounded on $\{\lambda: |\lambda|\leqq 1\}$ so that $\Theta_T(\lambda)$ satisfies in this case the requirements of our theorem.

Our method is the geometric interpretation of the characteristic function developed in [15, VI]. This interpretation is generalized to the case of operators which need not be contractions, by carrying forward the study of J-unitary dilation begun in [2]; but the proofs demand many considerations which did not arise for contractions. We include in § IV the proof of Sahnovič's theorem by our method.

We remark that our boundedness hypotheses are used in §§ III—IV only to ensure that we have a bounded operator on H^2, never to draw conclusions about the (operator) values which $\Theta_T(\lambda)$ assumes.

I. Preliminaries

1. As usual in this subject, it is important to note that the identity $T(I-T^*T)=(I-TT^*)T$ implies

(1. 1) $$Tf(I-T^*T)=f(I-TT^*)T$$

for any bounded complex Borel function f defined on the real line. In particular

(1. 2) $$TQ_T=Q_{T^*}T,\quad TJ_T=J_{T^*}T,$$

and taking adjoints,

(1. 2') $$Q_TT^*=T^*Q_{T^*},\quad J_TT^*=T^*J_{T^*}.$$

Thus $TJ_T Q_T = J_{T*} Q_{T*} T = Q_{T*} J_{T*} T$, which implies the fact already mentioned that

$$\Theta_T(\lambda)\mathfrak{D}_T \subseteqq \mathfrak{D}_{T*}. \tag{1.3}$$

For later use we derive another property of the characteristic function (cf. [6, § 2]). We begin with the fact, obvious from the definition (0. 1), that

$$\Theta_T(\lambda)^* = \Theta_{T*}(\bar{\lambda}) \tag{1.4}$$

whenever either side is defined. On the other hand,

$$\Theta_T(\lambda) Q_T J_T = Q_{T*}[-T+\lambda(I-\lambda T^*)^{-1}(I-T^*T)] = Q_{T*}(I-\lambda T^*)^{-1}(\lambda I - T).$$

Now assume that $\Theta_T(\lambda)$ and $\Theta_T(\bar{\lambda}^{-1})$ are both defined; we have

$$J_T\Theta_T(\bar{\lambda}^{-1})^* J_{T*}\Theta_T(\lambda) Q_T J_T = J_T \Theta_{T*}(\lambda^{-1}) J_{T*} Q_{T*}(I-\lambda T^*)^{-1}(\lambda I - T) =$$
$$= J_T Q_T (I-\lambda^{-1}T)^{-1}(\lambda^{-1}I - T^*)(I-\lambda T^*)^{-1}(\lambda I - T) = Q_T J_T,$$

from which it is easy to conclude that

$$\Theta_T(\lambda)^{-1} = J_T \Theta_T(\bar{\lambda}^{-1})^* J_{T*}|\mathfrak{D}_{T*}. \tag{1.5}$$

In particular, *if $\Theta_T(\lambda)$ is defined and bounded on $\{\lambda\colon |\lambda| \neq 1\}$ then $\Theta_T(\lambda)^{-1}$ exists and is bounded on* $D = \{\lambda\colon |\lambda| < 1\}$; thus in this case

$$\sup_D \|\Theta_T(\lambda)\| < \infty, \quad \sup_D \|\Theta_T(\lambda)^{-1}\| < \infty. \tag{1.6}$$

2. We now recall the construction of the J-unitary dilation [2]. The present discussion differs somewhat in notation, and deals only with bounded T.

The dilation will be an operator on a direct sum space

$$\mathfrak{K} = \cdots \oplus \overset{(-1)}{\mathfrak{D}_{T*}} \oplus \overset{(0)}{\mathfrak{H}} \oplus \overset{(1)}{\mathfrak{D}_T} \oplus \overset{(2)}{\mathfrak{D}_T} \oplus \cdots. \tag{1.7}$$

This means that there are canonical injections of $\mathfrak{H}$, $\mathfrak{D}_{T*}$, and $\mathfrak{D}_T$ onto orthogonal subspaces of $\mathfrak{K}$. We indicate these injections by supercript indices. Thus for any $h \in \mathfrak{H}$, $\overset{(0)}{h}$ denotes the corresponding element of the 0-th co-ordinate subspace $\overset{(0)}{\mathfrak{H}}$ of $\mathfrak{K}$; for any $h \in \mathfrak{D}_{T*}$, $\overset{(i)}{h}$ denotes the corresponding element of the i-th co-ordinate subspace $\overset{(i)}{\mathfrak{D}_{T*}}$ of $\mathfrak{K}$ ($i = -1, -2, \ldots$); and for any $h \in \mathfrak{D}_T$, $\overset{(i)}{h}$ denotes the corresponding element of the i-th co-ordinate subspace $\overset{(i)}{\mathfrak{D}_T}$ of $\mathfrak{K}$ ($i = 1, 2, \ldots$). The general element of $\mathfrak{K}$ is a sequence $\sigma = (h_i)_{i=-\infty}^{\infty}$ with $h_0 \in \mathfrak{H}$, $h_i \in \mathfrak{D}_{T*}$ $(i<0)$, $h_i \in \mathfrak{D}_T$ $(i>0)$, and $\|\sigma\|^2 = \sum_{i=-\infty}^{\infty} \|h_i\|^2 < \infty$; we can equally well write σ as a sum

$$\sigma = \sum_{i=-\infty}^{\infty} \overset{(i)}{h_i} \tag{1.8}$$

of elements in co-ordinate subspaces of $\mathfrak{K}$.

We define operators U and J on $\mathfrak{K}$ by specifying how they act on the above general element σ:

$$U\sigma = \sum_{i\neq 0,\,1} \overset{(i)}{h_{i-1}} + \overset{(0)}{h'} + \overset{(1)}{h''}, \tag{1.9}$$

$$h' = Q_{T^*}h_{-1} + Th_0, \quad h'' = -T^* J_{T^*} h_{-1} + Q_T h_0; \tag{1.9'}$$

$$J\sigma = \sum_{i<0} \overset{(i)}{(J_{T^*}h_i)} + \overset{(0)}{h_0} + \sum_{i>0} \overset{(i)}{(J_T h_i)}. \tag{1.10}$$

Then $J^* = J = J^{-1}$, and U is J-unitary, i.e.,

$$(JU\sigma, U\sigma') = (J\sigma, \sigma') \qquad (\sigma, \sigma' \in \mathfrak{K}) \tag{1.11}$$

and U is invertible; we shall have need for the explicit expression for its inverse, acting upon the general σ of (1.8):

$$U^{-1}\sigma = \sum_{i\neq -1,\,0} \overset{(i)}{h_{i+1}} + \overset{(-1)}{k'} + \overset{(0)}{k''}, \tag{1.12}$$

$$k' = J_{T^*}Q_{T^*}h_0 - J_{T^*}Th_1, \quad k'' = T^*h_0 + J_T Q_T h_1. \tag{1.12'}$$

U is a dilation of T, that is, for all $h \in \mathfrak{H}$,

$$\overset{(0)}{(T^n h)} = PU^n \overset{(0)}{h} \qquad (n = 0, 1, 2, \ldots), \tag{1.13}$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\overset{(0)}{\mathfrak{H}}$. We obtain a J-isometric dilation of T (i.e., an operator satisfying the analogues of (1.11) and (1.13), but not necessarily invertible) if we consider the restriction U_+ of U to a certain invariant subspace $\mathfrak{K}_+$. Namely, $\mathfrak{K}_+ = \bigvee_{n\geq 0} U^n \overset{(0)}{\mathfrak{H}}$, or, perhaps more simply, $\mathfrak{K}_+$ is the set of all vectors $\sum_{i=0}^{\infty} \overset{(i)}{h_i}$ in $\mathfrak{K}$. Evidently $\mathfrak{K}_+$ reduces J.

3. We conclude the preliminaries by recalling some well-known simple notions about geometry of subspaces of Hilbert spaces and J-spaces, which are central to our main arguments below. These will be stated in a general context: Let $\mathfrak{M}$ and $\mathfrak{N}$ be any subspaces of any Hilbert space $\mathfrak{H}$, and let P and Q respectively be the orthoprojectors onto $\mathfrak{M}$ and $\mathfrak{N}$. Then we have (see e.g. [1])

Lemma 1.1. *The operators $PQ|\mathfrak{M}$ and $QP|\mathfrak{N}$ have the same spectrum, except perhaps for* 0.

Let us say that $\mathfrak{M}$ is "not far from" $\mathfrak{N}$ in case $0 \notin \sigma(PQ|\mathfrak{M})$. (In more conventional terminology [3, 1], $\mathfrak{M}$ neither intersects nor is asymptotic to $\mathfrak{H} \ominus \mathfrak{N}$.) If A denotes $Q|\mathfrak{M}$ as an operator from $\mathfrak{M}$ to $\mathfrak{N}$, then $A^*A = PQ|\mathfrak{M}$; thus $\mathfrak{M}$ is not far from $\mathfrak{N}$ if and only if there exists $c > 0$ such that, for all $m \in \mathfrak{M}$, $\|Qm\| \geq c\|m\|$. A necessary and sufficient condition that $\mathfrak{M}$ be not far from $\mathfrak{N}$ and $\mathfrak{N}$ not far from $\mathfrak{M}$ is that $Q|\mathfrak{M}$ be an invertible map of $\mathfrak{M}$ onto $\mathfrak{N}$.

Lemma 1. 2. *If $\mathfrak{M}$ is not far from $\mathfrak{N}$ then $\mathfrak{H}\ominus\mathfrak{N}$ is not far from $\mathfrak{H}\ominus\mathfrak{M}$.*

This follows immediately from the previous Lemma: $0\notin\sigma(PQ|\mathfrak{M})$ implies $1\notin\sigma(P(1-Q)|\mathfrak{M})$, which implies $1\notin\sigma((1-Q)P|\mathfrak{H}\ominus\mathfrak{N})$, which implies

$$0\notin\sigma((1-Q)(1-P)|\mathfrak{H}\ominus\mathfrak{N}), \text{ q.e.d.}$$

See also [14, Lemma 9. 1. 1].

Lemma 1. 3. *If $\mathfrak{M}$ is not far from $\mathfrak{N}$ then $\mathfrak{M}+(\mathfrak{H}\ominus\mathfrak{N})$ is closed (and is the direct sum of $\mathfrak{M}$ and $\mathfrak{H}\ominus\mathfrak{N}$).*

This is well known, e.g. [9, § 3], [3, I].

Now let there also be defined on $\mathfrak{H}$ a symmetry J, i.e. $J^{-1}=J=J^*$, making it a J-space. We will use the notion of a regular subspace (pravil'noe podprostranstvo) of $\mathfrak{H}$ [5]. Let $\mathfrak{M}$ and P be as above; let P_+ denote $\frac{1}{2}(I+J)$, the orthoprojector onto the canonical positive subspace of $\mathfrak{H}$. $\mathfrak{M}$ is called "regular" in case it is not far from $J\mathfrak{M}$, in the sense defined above.

Using the fact that the orthoprojector onto $J\mathfrak{M}$ is JPJ, and that $PJPJ|\mathfrak{M}$ is the square of the hermitian operator $PJ|\mathfrak{M}$, it is not hard to see that each of the following conditions is equivalent to $\mathfrak{M}$ being regular:

(i) $\|PJx\|$ defines on $\mathfrak{M}$ a norm equivalent to the given norm;
(ii) $PJ|\mathfrak{M}$ has a bounded inverse on $\mathfrak{M}$;
(iii) $\frac{1}{2}\notin\sigma(PP_+|\mathfrak{M})$;
(iv) $\frac{1}{2}\notin\sigma(P_+P|P_+\mathfrak{H})$.

The equivalence of (iii) with (iv) here is a case of Lemma 1. 1.

Lemma 1. 4. *If $\mathfrak{M}$ is regular, then the following are also regular: $J\mathfrak{M}$; the orthogonal complement $\mathfrak{H}\ominus\mathfrak{M}$ of $\mathfrak{M}$; and the J-orthogonal complement $\mathfrak{H}\ominus J\mathfrak{M}$ of M.*

As to $J\mathfrak{M}$, this follows from (i) and the fact that J is unitary; as to $\mathfrak{H}\ominus\mathfrak{M}$, it follows from (iv); the rest is obvious.

It is only for regular subspaces that the J-orthogonal complement deserves its name:

Lemma 1. 5. *If $\mathfrak{M}$ is regular, then $\mathfrak{H}$ is the direct sum of $\mathfrak{M}$ and $\mathfrak{H}\ominus J\mathfrak{M}$.*

This is a corollary of Lemma 1. 3. (The converse is known too, but we will not need it.)

We now return to the special context of the Introduction, so the symbols $\mathfrak{H}$, J, etc. will have the special meanings which were attached to them.

II. The characteristic function and the J-unitary dilation

1. We will now show that the dilation contruction gives rise to the characteristic function here in almost as natural a way as in the case of contractions.

For this purpose we consider two subspaces on which U_+ acts as a unilateral shift (of some multiplicity $\geqq 0$). First,

$$\mathfrak{R}_+ = \overset{(0)}{\mathfrak{H}} \oplus \mathfrak{M}, \quad \mathfrak{M} = \bigoplus_{i=1}^{\infty} \overset{(i)}{\mathfrak{D}_T} = \bigvee_{n \geqq 0} U^n \overset{(1)}{\mathfrak{D}_T}, \tag{2.1}$$

and $U_+|\mathfrak{M}$ is, by definition, an isometric mapping of each co-ordinate subspace onto the next.

Second, we consider

$$\mathfrak{M}_* = \bigvee_{n \geqq 0} U^{n+1} \overset{(-1)}{\mathfrak{D}_{T^*}}. \tag{2.2}$$

It is plain from (1.9), (1.9′) that $\mathfrak{M}_* \subseteqq \mathfrak{R}_+$. In the contractive case, it was shown [15] that in (2.2) as well, U_+ maps each of the sequence of subspaces isometrically onto the next. In the general case, it need not be isometric, but it is expansive: for all

$$\sigma = \sum_{i=0}^{\infty} \overset{(i)}{h_i} \in \mathfrak{R}_+$$

we have

$$\|U_+\sigma\|^2 = \|Th_0\|^2 + \|Q_T h_0\|^2 + \|h_1\|^2 + \cdots \geqq$$

$$\geqq \|Th_0\|^2 + (J_T Q_T h_0, Q_T h_0) + \|h_1\|^2 + \cdots = \|h_0\|^2 + \|h_1\|^2 + \cdots = \|\sigma\|^2.$$

2. Let us now introduce the Fourier representations of $\mathfrak{M}$ and $\mathfrak{M}_*$. For finite sums

$$\sigma = \sum_{n=0}^{N} \overset{(n+1)}{h_n} = \sum_{n=0}^{N} U^n \overset{(1)}{h_n} \in \mathfrak{M}, \qquad \sigma_* = \sum_{n=0}^{N_*} U^{n+1} \overset{(-1)}{h_{*n}} \in \mathfrak{M}_* \tag{2.3}$$

(where $h_n \in \mathfrak{D}_T$, $h_{*n} \in \mathfrak{D}_{T^*}$), we put

$$\Phi\sigma(\lambda) = \sum_{n=0}^{N} \lambda^n h_n, \quad F\sigma_*(\lambda) = \sum_{n=0}^{N_*} \lambda^n h_{*n} \quad (|\lambda| < 1). \tag{2.4}$$

Linear applications are thereby defined from dense subsets of $\mathfrak{M}$, resp. $\mathfrak{M}_*$, into the space $H^2(\mathfrak{D}_T)$, resp. $H^2(\mathfrak{D}_{T^*})$. These are Hardy H^2 spaces of vector-valued functions, see [15, V]. The mapping Φ is obviously isometric and can be extended to a unitary mapping of $\mathfrak{M}$ onto $H^2(\mathfrak{D}_T)$, which will still be denoted by Φ. Under this isomorphism, the isometric unilateral shift $U_+|\mathfrak{M}$ corresponds to Λ: $\Phi U_+|\mathfrak{M} = \Lambda\Phi$. Here Λ is the multiplication by the independent variable, that is, for $u \in H^2(\mathfrak{D}_T)$ we have $\Lambda u(\lambda) = \lambda u(\lambda)$ ($|\lambda| < 1$). This correspondence of unilateral shift to multiplication is the essential feature of the Fourier representation. It carries over to the non-isometric Fourier representation F: if Λ_* denotes the multiplication by λ in $H^2(\mathfrak{D}_{T^*})$ then obviously $FU\sigma_* = \Lambda_* F\sigma_*$ for the above finite sums σ_*.

We introduce J-space structure in the H^2 spaces in the natural way. Denote by $\mathbf{J}$ the operator defined on $H^2(\mathfrak{D}_T)$ by $(\mathbf{J}u)(\lambda) = J_T(u(\lambda))$ $(|\lambda|<1)$. It is immediate that $\Phi J|\mathfrak{M} = \mathbf{J}\Phi$ and hence identically $(\mathbf{J}\Phi\sigma, \Phi\sigma) = (J\sigma, \sigma)$ $(\sigma \in \mathfrak{M})$, showing how to regard Φ as preserving also the J-space structure. Similarly, define $\mathbf{J}_*$ on $H^2(\mathfrak{D}_{T^*})$ by $(\mathbf{J}_* u_*)(\lambda) = J_{T^*}(u_*(\lambda))$. We will verify the relation

$$(\mathbf{J}_* F\sigma_*, F\sigma_*) = (J\sigma_*, \sigma_*) \tag{2.5}$$

for finite sums in $\mathfrak{M}_*$, but it is less immediate because the terms in the definition (2.3) of σ_* do not belong to subspaces which are clearly invariant under J. However, the J-unitary property (1.11) of U allows us to write

$$(J\sigma_*, \sigma_*) = \sum_{n=0}^{N_*} \sum_{m=0}^{N_*} (JU^{n+1} \overset{(-1)}{h}_{*n}, U^{m+1} \overset{(-1)}{h}_{*m}) = \sum_{n=0}^{N_*} (J \overset{(-1)}{h}_{*n}, \overset{(-1)}{h}_{*n}) = \sum_{n=0}^{N_*} (J_{T^*} h_{*n}, h_{*n})$$

(the terms for $m \neq n$ vanish because $U^{m-n} \overset{(-1)}{\mathfrak{D}}_{T^*} \perp \overset{(-1)}{\mathfrak{D}}_{T^*}$). But the right-hand member, by the definition of J_* and the definition of the inner product in H^2, is equal to $(\mathbf{J}_* F\sigma_*, F\sigma_*)$, with $F\sigma_*$ as in (2.4). Thus (2.5) is proved.

3. We thus have two naturally defined subspaces $\mathfrak{M}$ and $\mathfrak{M}_*$, and the projectors $P_{\mathfrak{M}}$, $P_{\mathfrak{M}^*}$ onto them do not commute. It is not surprising that fairly complete information about T is contained in an invariant description of the contraction $P_{\mathfrak{M}^*}|\mathfrak{M}$. If one tries to make this description giving $\mathfrak{M}$ and $\mathfrak{M}_*$ their Fourier representations, one finds the contraction from $\mathfrak{M}$ to $\mathfrak{M}_*$ is replaced by a mapping from $H^2(\mathfrak{D}_T)$ to $H^2(\mathfrak{D}_{T^*})$, given exactly by the characteristic function.

We now exhibit this relationship formally, for arbitrary T. In the following section we will give it a geometric sense, by using the hypothesis of boundedness of the characteristic function.

For any $u \in H^2(\mathfrak{D}_T)$, with power-series expansion $u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$, let $\Theta_T u$ denote the function whose values are defined by $(\Theta_T u)(\lambda) = \Theta_T(\lambda) u(\lambda)$. This function is defined and analytic, with values in $\mathfrak{D}_{T^*}$, at least for $|\lambda| < \min(\|T\|^{-1}, 1)$, and this is all we need for the moment (indeed it would be possible to proceed using only formal power series). We can write in the neighborhood of $\lambda = 0$

$$(\Theta_T u)(\lambda) = \sum_{n=0}^{\infty} \lambda^n \left(\sum_{m=0}^{n} \theta_{n-m} u_m \right); \tag{2.6}$$

here the θ_n are the Taylor coefficients of Θ_T: $\Theta_T(\lambda) = \sum_{n=0}^{\infty} \lambda^n \theta_n$. From the definition

(0. 1) we can derive this explicit expression:

$$(2.7)\qquad (JU^m \overset{(1)}{h}, U^{n+1} \overset{(-1)}{h_*}) = \begin{cases} 0 & \text{if } n<m, \\ (\theta_{n-m} J_T h, h_*) & \text{if } n \geqq m \end{cases}$$

for all $h \in \mathfrak{D}_T$, $h_* \in \mathfrak{D}_{T^*}$. To prove this, use the J-unitary property in the same way as above:

$$(JU^m \overset{(1)}{h}, U^{n+1} \overset{(-1)}{h_*}) = (JU^{m-n-1} \overset{(1)}{h}, \overset{(-1)}{h_*})$$

which obviously is 0 for $m \geqq n+1$. If $m-n-1 = -k$, $k>0$, then we need to find the component of $U^{-k} \overset{(1)}{h}$ in $\overset{(-1)}{\mathfrak{D}}_{T^*}$; this we can do by iterating (1. 12), (1. 12′), and the result is

$$-J_{T^*}Th \quad \text{if} \quad k=1, \qquad J_{T^*}Q_{T^*}T^{*k-2}J_TQ_Th \quad \text{if} \quad k>1.$$

Therefore

$$(JU^{-k} \overset{(1)}{h}, \overset{(-1)}{h_*}) = \begin{cases} (-Th, h_*) = (\theta_0 J_T h, h_*) & \text{if } k=1, \\ (Q_{T^*}T^{*k-2}Q_TJ_Th, h_*) = (\theta_{k-1}J_Th, h_*) & \text{if } k>1 \end{cases}$$

using (0. 1). This establishes (2. 7).

We are now in a position to discuss inner products of elements of $\mathfrak{M}$ with elements of $\mathfrak{M}_*$. Let σ, σ_* be as in (2. 3). Then, by (2. 7),

$$(2.8)\qquad (J\sigma, \sigma_*) = \sum_{m=0}^{N} \sum_{n=0}^{N_*} (JU^m \overset{(1)}{h_m}, U^{n+1} \overset{(-1)}{h_{*n}}) = \sum_{n \geqq m \geqq 0} (\theta_{n-m} J_T h_m, h_{*n}).$$

Also by (2. 6) and (2. 4)

$$(\Theta_T \Phi J\sigma)(\lambda) = (\Theta_T J_T \Phi\sigma)(\lambda) = \sum_{n=0}^{\infty} \lambda^n \left(\sum_{m=0}^{\min(n,N)} \theta_{n-m} J_T h_m \right).$$

This is analitic in λ with values in $\mathfrak{D}_{T^*}$, but need not lie in $H^2(\mathfrak{D}_{T^*})$; if it does, its inner product with $F\sigma_*$ from (2. 4) is, by (2. 8),

$$\sum_{n=0}^{N_*} \left(\sum_{m=0}^{\min(n,N)} \theta_{n-m} J_T h_m, h_{*n} \right) = (J\sigma, \sigma_*).$$

III. Geometric properties of the J-unitary dilation in case the characteristic function is bounded

1. Assume now that Θ_T is defined on the open unit disk D and that

$$\sup_D \|\Theta_T(\lambda)\| = C < \infty.$$

Then for any $u \in H^2(\mathfrak{D}_T)$, $\Theta_T u$ belongs to $H^2(\mathfrak{D}_{T^*})$ and its norm in that space is $\leqq C\|u\|$. Let $\Theta: H^2(\mathfrak{D}_T) \to H^2(\mathfrak{D}_{T^*})$ be defined by $\Theta u = \Theta_T u$; it is an operator of norm C.

The conclusion of the last section can now be rewritten as

$$(J\sigma, \sigma_*) = (\Theta\Phi J\sigma, F\sigma_*).$$

Because elements $J\sigma$ (with σ a finite sum (2.3)) are dense in $\mathfrak{M}$, and Θ and Φ are continuous, we deduce that

$$(3.1) \qquad (\mu, \sigma_*) = (\Theta\Phi\mu, F\sigma_*) \qquad (\mu \in \mathfrak{M}).$$

This is not quite the promised interpretation of Θ_T in terms of $P_{\mathfrak{M}^*}$ because the second factor in the inner product is still restricted to be a finite sum.

We will remedy this by proving *that F has a unique extension to an affinity of $\mathfrak{M}_*$ onto $H^2(\mathfrak{D}_{T^*})$.*

To this end taking $\mu = P_{\mathfrak{M}}\sigma_*$ in (3.1), we obtain

$$\|P_{\mathfrak{M}}\sigma_*\|^2 = (P_{\mathfrak{M}}\sigma_*, \sigma_*) = (\Theta\Phi P_{\mathfrak{M}}\sigma_*, F\sigma_*) \leqq C\|P_{\mathfrak{M}}\sigma_*\| \, \|F\sigma_*\|$$

whence

$$(3.2) \qquad \|P_{\mathfrak{M}}\sigma_*\|^2 \leqq C^2\|F\sigma_*\|^2.$$

Let P denote the projection onto the complement of $\mathfrak{M}$ in $\mathfrak{K}_+$, which by (2.1) is $\overset{(0)}{\mathfrak{H}}$. By (2.5) and the definition (1.10) of J,

$$(\mathbf{J}_* F\sigma_*, F\sigma_*) = (J\sigma_*, \sigma_*) = \|P\sigma_*\|^2 + (JP_{\mathfrak{M}}\sigma_*, P_{\mathfrak{M}}\sigma_*),$$

which yields, because $\mathbf{J}_*$ and J are contractions,

$$\|P\sigma_*\|^2 \leqq \|P_{\mathfrak{M}}\sigma_*\|^2 + \|F\sigma_*\|^2 \leqq (C^2+1)\|F\sigma_*\|^2$$

(using (3.2)). Add this to (3.2) to obtain

$$\|\sigma_*\|^2 \leqq (1+2C^2)\|F\sigma_*\|^2.$$

This proves that F has a bounded inverse G. The domain of G is dense, so G has a unique bounded extension to the whole of $H^2(\mathfrak{D}_{T^*})$; denote this also by G. By continuity, we deduce from (2.5) that

$$(JGu, Gu) = (\mathbf{J}_* u, u) \qquad (u \in H^2(\mathfrak{D}_{T^*})).$$

This is the same as saying $G^* P_{\mathfrak{M}_*} JG = \mathbf{J}_*$ $(= \mathbf{J}_*^{-1})$. Thus G has the left-inverse $\mathbf{J}_* G^* P_{\mathfrak{M}^*} J$, which as a product of bounded operators is bounded. It is an extension of F because F is inverse to G on a dense set. This completes the proof that F has a unique extension to an affinity; the extension will still be denoted by F.

Then we know also that σ_* in (3.1) can be replaced by an arbitrary element μ_* of $\mathfrak{M}_*$.

2. We now introduce the residual part of U, in imitation of the contraction case. The images of $\overset{(-1)}{\mathfrak{D}}_{T^*}$ under non-negative powers of U^{-1} do span all of $\mathfrak{K} \ominus \mathfrak{K}_+$.

Its images under positive powers of U, on the other hand, span the subspace $\mathfrak{M}_*$ which need not be all of $\mathfrak{K}_+$. Consider the J-orthogonal complement of $\mathfrak{M}_*$:

$$\mathfrak{R} = \mathfrak{K}_+ \ominus J\mathfrak{M}_* = \mathfrak{K} \ominus J \bigvee_{-\infty}^{\infty} U^n \overset{(-1)}{\mathfrak{D}}_{T^*}; \tag{3. 3}$$

it is clear that the two definitions are equivalent. The latter expression (3. 3), together with the J-unitary property of U (1. 11), make it clear that $\mathfrak{R}$ is invariant under both U and U^{-1}. Thus we may define the "residual part" $R = U|\mathfrak{R}$, an invertible operator.

Being a restriction of U_+ (not just of U), R is expansive (see § II.1 above). Hence

$$\|R^{-1}\| \leqq 1. \tag{3. 4}$$

Our next aim is to prove that

$$\sup_{-\infty<n<\infty} \|R^n\| < \infty, \tag{3. 5}$$

and (3. 4) takes care of this for all $n \leqq 0$.

Return to (2. 5), which implies at once

$$F^* \mathbf{J}_* F = P_{\mathfrak{M}^*} J | \mathfrak{M}_*.$$

Now that we are able to assert that F (and therefore also F^*) is an affinity, we can deduce that the equation represents an invertible operator on $\mathfrak{M}_*$. That is, $\mathfrak{M}_*$ is a regular subspace of the J-space $\mathfrak{K}_+$. (See § I. 3.) By Lemma 1. 4, we deduce now from (3. 3) that $\mathfrak{R}$ is also regular, that is, that $P_{\mathfrak{R}} J|\mathfrak{R}$ is invertible.

Set $J_{\mathfrak{R}} = P_{\mathfrak{R}} J|\mathfrak{R}$. We now know that for some $c > 0$

$$c\|\varrho\| \leqq \|J_{\mathfrak{R}}\varrho\| \leqq \|\varrho\| \qquad (\varrho \in \mathfrak{R}). \tag{3. 6}$$

But we also know from the remarks following (3. 3) that

$$(J_{\mathfrak{R}} R^{-1}\varrho, R^{-1}\varrho') = (JU^{-1}\varrho, U^{-1}\varrho') = (J\varrho, \varrho') = (J_{\mathfrak{R}}\varrho, \varrho')$$

for $\varrho, \varrho' \in \mathfrak{R}$, so that (iterating) $J_{\mathfrak{R}} = (R^{-n})^* J_{\mathfrak{R}} R^{-n}$ $(n > 0)$. With (3. 4) and (3. 6), this gives

$$c\|\varrho\| \leqq \|J_{\mathfrak{R}}\varrho\| = \|(R^{-n})^* J_{\mathfrak{R}} R^{-n}\varrho\| \leqq \|J_{\mathfrak{R}} R^{-n}\varrho\| \leqq \|R^{-n}\varrho\|,$$

whence $\|R^n\varrho\| \leqq \frac{1}{c}\|\varrho\|$ $(\varrho \in \mathfrak{R};\ n = 1, 2, \ldots)$.

To sum up, (3. 5) has been established, with

$$\sup_{n>0} \|R^n\| \leqq \frac{1}{c}; \quad \sup_{n<0} \|R^n\| \leqq 1.$$

Now we appeal to the theorem of B. Sz.-Nagy that any operator R with

$\sup\limits_{-\infty<n<\infty} \|R^n\| \leqq \frac{1}{c} < \infty$ is similar to a unitary [14]. More precisely, it tells us that *there exists a self-adjoint invertible operator A on $\mathfrak{R}$ such that*

$$\|A\| \cdot \|A^{-1}\| \leqq \frac{1}{c}$$

and such that $V = A^{-1}RA$ is unitary.

3. We are ready to prove the theorem stated in the introduction. We begin by defining a new Hilbert space

$$\mathbf{H} = H^2(\mathfrak{D}_{T^*}) \oplus \mathfrak{R}$$

with a canonical mapping into $\mathfrak{K}_+$:

$$X(u \oplus \varrho) = F^{-1}u + A\varrho \qquad (u \in H^2(\mathfrak{D}_{T^*}),\ \varrho \in \mathfrak{R}). \tag{3.7}$$

As u and ϱ vary, the term $F^{-1}u$ here ranges over all of $\mathfrak{M}_*$ and the term $A\varrho$ over all of $\mathfrak{R}$, because F^{-1} and A are affinities. But $\mathfrak{R}$ is the J-orthogonal complement of $\mathfrak{M}_*$ by definition (3. 3), and $\mathfrak{M}_*$ was just proved to be regular, so by Lemma 1. 5, X maps $\mathbf{H}$ onto $\mathfrak{K}_+$.

Let P again denote the orthoprojector on $\mathfrak{K}_+$ onto $\overset{(0)}{\mathfrak{H}}$; we now see that PX maps $\mathbf{H}$ onto $\overset{(0)}{\mathfrak{H}}$. Let Q denote the orthoprojector on $\mathbf{H}$ onto the orthogonal complement of the null-space of PX. We define Y: $Q\mathbf{H} \to \mathfrak{H}$ by

$$Y(u \oplus \varrho) = h \text{ if and only if } PX(u \oplus \varrho) = \overset{(0)}{h}. \tag{3.8}$$

Being continuous, 1—1, and onto, Y must be an affinity of $Q\mathbf{H}$ onto $\mathfrak{H}$.

Now the operator $\mathbf{U}$ defined by

$$\mathbf{U}(u \oplus \varrho) = \Lambda_* u \oplus V\varrho,$$

where V is the unitary found in § III. 2, is an isometry on $\mathbf{H}$; and it is related to U_+ by the application (3. 7):

$$X\mathbf{U} = F^{-1}\Lambda_* + AV = U_+ F^{-1} + RA = U_+ X. \tag{3.9}$$

We project down onto $\overset{(0)}{\mathfrak{H}}$. That is, we operate on (3. 9) on the left by P; using the definition (3. 8) and the dilation property (1. 13), we obtain

$$YQ\mathbf{U} = TY.$$

But Y is an affinity and $Q\mathbf{U}$ is certainly a contraction (on $Q\mathbf{H}$ to $Q\mathbf{H}$). This completes the proof of the theorem.

IV. Similarity to a unitary operator

This section will be devoted to the proof of the result of SAHNOVIČ stated in the introduction. Accordingly we now strengthen the hypotheses used in § III, and assume that $\Theta_T(\lambda)$ is defined for $|\lambda| \neq 1$ and

$$\sup_{|\lambda|\neq 1} \|\Theta_T(\lambda)\| = C < \infty.$$

We saw in § I. 1 that this makes $\Theta_T(\lambda)$ and $\Theta_T(\lambda)^{-1}$ both uniformly bounded analytic operator-functions on D, see (1. 5) and (1. 6). Therefore Θ is an affinity of $H^2(\mathfrak{D}_T)$ onto $H^2(\mathfrak{D}_{T^*})$; indeed its inverse is given by

$$(\Theta^{-1}u_*)(\lambda) = \Theta_T(\lambda)^{-1}u_*(\lambda) \qquad (|\lambda| < 1)$$

for $u_* \in H^2(\mathfrak{D}_{T^*})$.

We begin, as before, with (3. 1), extended to

$$(\mu, \mu_*) = (\Theta\Phi\mu, F\mu_*) \qquad (\mu \in \mathfrak{M},\ \mu_* \in \mathfrak{M}_*),$$

$$P_{\mathfrak{M}^*}|\mathfrak{M} = F^*\Theta\Phi. \tag{4. 1}$$

Now, however, since all three operators on the right are affinities, we are able to short-cut the considerations of § III. 3. Indeed, (4. 1) says directly that $P_{\mathfrak{M}_*}|\mathfrak{M}$ is an affinity of $\mathfrak{M}$ onto $\mathfrak{M}_*$. This implies that $\mathfrak{M}$ is not far from $\mathfrak{M}_*$ and $\mathfrak{M}_*$ is not far from $\mathfrak{M}$, in the sense of § I. 3. By Lemma 1. 2, $\overset{(0)}{\mathfrak{H}}$ is not far from $J\mathfrak{R}$ and vice versa. Applying the unitary J, we see that $J\overset{(0)}{\mathfrak{H}}$ $(=\overset{(0)}{\mathfrak{H}})$ is in the same relationship to $\mathfrak{R}$. Hence $P|\mathfrak{R}$ is an affinity of $\mathfrak{R}$ onto $\overset{(0)}{\mathfrak{H}}$ just as in the contraction case.

Let A, V be the operators found in § III. 2. Define $Y\colon\ \mathfrak{R} \to \mathfrak{H}$ by

$$Y\varrho = h \quad \text{if and only if} \quad PA\varrho = \overset{(0)}{h}.$$

Then Y is an affinity from $\mathfrak{R}$ onto $\mathfrak{H}$; and the equation

$$PAV = PRA = PUA,$$

together with the dilation relation (1. 13), gives $YV = TY$. This is a similarity of T to a unitary operator, as was required.

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О характеристических функциях обратимого оператора

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Начиная с известных работ М. С. Лившица [1, 2], понятие характеристической функции стало играть фундаментальную роль при исследовании спектральных свойств операторов. Это понятие претерпело большую эволюцию, которая еще, возможно, не закончилась. По-видимому, в определенном смысле можно считать окончательным определение характеристической функции (W-функции) оператора, «близкого» к самосопряженному, введенное М. С. Бродским [3, 4].

В настоящей статье дается подробное обоснование результатов [5], которые, по мнению авторов, должны способствовать стабилизации нового определения характеристической функции (Θ-функции) обратимого оператора «близкого» к унитарному. Это определение удовлетворяет следующим естественным требованиям: 1. Θ-функция удовлетворяет фундаментальным тождествам и неравенствам (см. § 2). 2. Θ-функция выдерживает дробно-линейные преобразования. При этом обнаруживается ее прямая связь с W-функцией (см. § 3). 3. По Θ-функции с необходимой точностью восстанавливается оператор (и более того — соответствующий узел) (§ 6). 4. Θ-функция допускает полную внутреннюю аналитическую характеристику (§§ 4, 5, 6). 5. Для Θ-функции имеет место теорема умножения [6, 7]. 6. При весьма общих условиях относительно основного оператора Θ-функция допускает мультипликативное представление (см. § 7 и [8]).

Существенно также, что выработанные ранее определения характеристической функции (см. [1, 2, 9—14]) укладываются в общую схему определения Θ-функции (§ 1). Наконец, добавим, что Θ-функция уже нашла важные приложения в работах Л. А. Сахновича [15, 16][1]), а совсем недавно обнаружилось, что она играет существенную роль также в теории резольвентных матриц эрмитовых операторов [17].

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[1]) В этих работах, по существу, используется определение характеристической функции обратимого оператора, несущественно отличающееся от нашего определения Θ-функции.

§ 1. Определение характеристической функции

Как известно [18, 4], $\mathcal{W}$*-узлом* для основного оператора $A \in [\mathfrak{H}, \mathfrak{H}]$[2]) называется совокупность гильбертовых пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов $A \in [\mathfrak{H}, \mathfrak{H}]$, $S \in [\mathfrak{G}, \mathfrak{H}]$, $J \in [\mathfrak{G}, \mathfrak{G}]$, таких что

$$\operatorname{Im} A = SJS^* \quad \left(\operatorname{Im} A = \frac{A-A^*}{2i}\right), \quad J^2 = I, J^* = J. \tag{1.1}$$

$\mathcal{W}$-узлу сопоставляется *характеристическая функция* (*W-функция*), определенная равенством[3])

$$W(\lambda) = I + 2iJS^*(A^* - \lambda I)^{-1}S \qquad (\bar{\lambda} \in \sigma(A)). \tag{1.2}$$

Эта функция была введена М. С. Бродским [3, 4] в обобщение определения W-характеристической функции, предложенного М. С. Лившицем [19]. Определение W-функции допускает естественное расширение на случай неограниченного оператора с ограниченной мнимой компонентной [20, 21].

Более сложным является понятие характеристической функции оператора «близкого» к унитарному (Θ-функции), которое сравнительно недавно появилось в различных исследованиях.

Пусть T — некоторый оператор из $[\mathfrak{H}, \mathfrak{H}]$. Всегда найдется гильбертово пространство $\mathfrak{G}$ и операторы $R \in [\mathfrak{G}, \mathfrak{H}]$ и $J \in [\mathfrak{G}, \mathfrak{G}]$ такие, что

$$I - T^*T = RJR^*, \quad J^2 = I, \quad J^* = J. \tag{1.3}$$

В самом деле, этого можно достигнуть, положив, например,

$$\mathfrak{G} = \mathfrak{D}_T (= \overline{(I - T^*T)\mathfrak{H}}), \quad J = \operatorname{sign}(I - T^*T)|\mathfrak{D}_T, \quad R = |I - T^*T|^{\frac{1}{2}}|\mathfrak{D}_T. \tag{1.4}$$

Если оператор T обратим ($T^{-1} \in [\mathfrak{H}, \mathfrak{H}]$), то совокупность пространств $\mathfrak{H}, \mathfrak{G}$ и операторов T, R, J связанных между собою соотношениями (1.3), назовем $\mathcal{Y}$*-узлом* и обозначим символом

$$\mathcal{Y} = (\mathfrak{H}, \mathfrak{G}; T, R, J). \tag{1.5}$$

При этом оператор T назовем *основным* оператором $\mathcal{Y}$-узла. Очевидно, что существует множество различных узлов с основным оператором T. Следует подчеркнуть, что даже при фиксированном $\mathfrak{G}$, если оно бесконечномерно, и фиксированном J, если оно индефинитно, возможно включение оператора T в различные $\mathcal{Y}$-узлы, в которых операторы R будут отличаться друг от друга на неограниченные правые «J-унитарные» множители.

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[2]) Если $\mathfrak{H}_1, \mathfrak{H}_2$ — гильбертовы пространства, то символом $[\mathfrak{H}_1, \mathfrak{H}_2]$ обозначается множество всех линейных ограниченных операторов, действующих из $\mathfrak{H}_1$ в $\mathfrak{H}_2$. Для $A \in [\mathfrak{H}, \mathfrak{H}]$ через $\sigma(A)$ обозначается спектр оператора A.

[3]) Некоторые авторы, в частности, М. С. Бродский записывают W-функцию в ином виде:

$$W(\lambda) = I - 2iS^*(A - \lambda I)^{-1}SJ \qquad (\lambda \bar{\in} \sigma(A)).$$

Лемма 1.1. *Для любого узла* (1.5) *найдется обратимый оператор* $K \in [\mathfrak{G}, \mathfrak{G}]$ *такой, что*

$$J - R^* R = K^* J K. \tag{1.6}$$

Если обратимый оператор $K \in [\mathfrak{G}, \mathfrak{G}]$ *удовлетворяет условию* (1.6), *то он представим в виде*

$$K = U_0 H_0, \tag{1.7}$$

где U_0, $H_0 \in [\mathfrak{G}, \mathfrak{G}]$ — *операторы, обладающие следующими свойствами*

$$U_0^* J U_0 = U_0 J U_0^* = J, \tag{1.8}$$

$$H_0^2 = I - JR^* R, \quad \sigma(H_0) \subset (0, \infty), \quad (JH_0)^* = JH_0. \tag{1.9}$$

Доказательство. Спектр оператора $RJR^* = I - T^*T$ лежит на открытом интервале $(-\infty, 1)$. Следовательно[4]), на этом же интервале расположен спектр оператора JR^*R. Таким образом, оператор $G = I - JR^*R$ обладает положительным спектром и обратим. Кроме того, G является J-самосопряженным оператором. В силу теоремы В. П. Потапова и Ю. П. Гинзбурга [23, 24] (см. также [25]) всякий обратимый J-самосопряженный оператор G с положительным спектром является квадратом некоторого обратимого J-самосопряженного оператора $H_0 = G^{\frac{1}{2}}$ с положительным спектром. Следовательно,

$$J - R^* R = JG = (JG^{\frac{1}{2}} G^{\frac{1}{2}})^* = (G^{\frac{1}{2}})^* (JG^{\frac{1}{2}})^* = (G^{\frac{1}{2}})^* J G^{\frac{1}{2}}$$

и можно положить $K = G^{\frac{1}{2}}$.[5])

Пусть обратимый оператор $K \in [\mathfrak{G}, \mathfrak{G}]$ удовлетворяет условию (1.6). Отсюда и из теоремы Потапова—Гинзбурга [23, 24] (см. также теоремы 4.3, 5.2 статьи [25]) вытекают формулы (1.8) и (1.9). Лемма доказана.

Характеристической функцией (*Θ-функцией*) *узла* (1.5) называется функция $\Theta_{\mathscr{Y}}(\zeta)$, определенная равенством

$$\Theta_{\mathscr{Y}}(\zeta) = J(\mathscr{K}^*)^{-1}(J - R^*(I - \zeta T^*)^{-1} R) \qquad (\zeta \,\bar{\in}\, \sigma[(T^*)^{-1}]), \tag{1.10}$$

где K — произвольный обратимый оператор из $[\mathfrak{G}, \mathfrak{G}]$, являющийся решением уравнения (1.6). Таким образом, *Θ-функция определяется с точностью до левого J-унитарного множителя* $U \in [\mathfrak{G}, \mathfrak{G}]$.

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[4]) Это утверждение вытекает из известного предложения [22] о том, что если оператор $BA - \lambda I$ $(\lambda \neq 0)$ обратим, то обратим оператор $AB - \lambda I$, причем

$$(AB - \lambda I)^{-1} = -\frac{1}{\lambda} I - \frac{1}{\lambda^2} AB + \frac{1}{\lambda^2}(BA - \lambda I)^{-1} BAB.$$

В дальнейшем мы неоднократно будем пользоваться соотношением $A(BA - \lambda I)^{-1} = (AB - \lambda I)^{-1} A$, которое вытекает из очевидного равенства $A(BA - \lambda I) = (AB - \lambda I)A$.

[5]) Доказательством этого предложения авторы обязаны Ю. Л. Шмульяну.

Мы не будем различать Θ-функции, отвечающие одному и тому же узлу.

Заметим, что $\Theta_{\mathscr{Y}}(0) = J(K^*)^{-1}(J-R^*R) = J(K^*)^{-1}K^*JK = K$, и поэтому

$$\Theta^*_{\mathscr{Y}}(0)J\Theta_{\mathscr{Y}}(\zeta) = J - R^*(I-\zeta T^*)^{-1}R. \tag{1.11}$$

Подчеркнем, что и обратно, *всякая оператор-функция* $\Theta(\zeta)$, *для которой выполняется равенство* $\Theta^*(0)J\Theta(\zeta) = J-R^*(I-\zeta T^*)^{-1}R$ *и оператор* $\Theta(0)\in[\mathfrak{G}, \mathfrak{G}]$ *обратим* $\left(\Theta^{-1}(0)\in[\mathfrak{G}, \mathfrak{G}]\right)$, *является некоторой характеристической функцией узла* $Y = (\mathfrak{H}, \mathfrak{G}; T, R, J)$.

Θ-функции $\mathscr{Y}$-узлов, в которых оператор T является основным, будем называть *Θ-функциями оператора* T.

Отметим, что в частном случае, когда $\mathfrak{G}$, R и J выбраны в соответствии с равенствами (1. 4), оператор K следует положить равным сужению $UT|\mathfrak{D}_T$, где U — какой-либо линейный ограниченный оператор, действующий из пространства $\mathfrak{D}_{T^*} = \overline{(I-TT^*)\mathfrak{H}}$ в пространство $\mathfrak{D}_T$ так, что

$$U^*JUJ_{T^*}, \quad UJ_{T^*}U^* = J \quad \left(J_{T^*} = \operatorname{sign}(I-TT^*)\,|\,\mathfrak{D}_{T^*}\right).$$

Тогда Θ-функция примет вид

$$\Theta_{\mathscr{Y}}(\zeta) = U\left(J_{T^*}(T^*)^{-1}J - J_{T^*}(T^*)^{-1}\,|I-T^*T|^{\frac{1}{2}}(I-\zeta T^*)^{-1}\,|I-T^*T|^{\frac{1}{2}}\right)\big|\,\mathfrak{D}_T. \tag{1.12}$$

Если отбросить множитель U и произвести некоторые простые преобразования, то придем к оператор-функции

$$\Theta_T(\zeta) = \left(T - \zeta J_{T^*}|I-TT^*|^{\frac{1}{2}}(I-\zeta T^*)^{-1}\,|I-T^*T|^{\frac{1}{2}}\right)\big|\,\mathfrak{D}_T. \tag{1.13}$$

Значения этой функции принадлежат $[\mathfrak{D}_T, \mathfrak{D}_{T^*}]$. Функция (1. 13) имеет смысл для любых операторов $T\in[\mathfrak{H}, \mathfrak{H}]$.

Поясним, как из (1. 12) получить (1. 13). Ввиду равенств

$$(T^*)^{-1}\,|I-T^*T|^{\frac{1}{2}} = |I-TT^*|^{\frac{1}{2}}(T^*)^{-1}$$

и

$$(T^*)^{-1}(I-\zeta T^*)^{-1} = (T^*)^{-1} + \zeta(I-\zeta T^*)^{-1}$$

имеем по отбрасывании множителя $\mathscr{U}$

$$\begin{aligned}\Theta_T(\zeta) = \big(J_{T^*}(T^*)^{-1}J - J_{T^*}(T^*)^{-1}\,|I-T^*T| - {}\\ - \zeta J_{T^*}|I-TT^*|^{\frac{1}{2}}(I-\zeta T^*)^{-1}\,|I-T^*T|^{\frac{1}{2}}\big)\big|\,\mathfrak{D}_T.\end{aligned} \tag{1.14}$$

Так как $(T^*)^{-1}J = J_{T^*}(T^*)^{-1}\,|\,\mathfrak{D}_T$, то

$$J_{T^*}(T^*)^{-1}\,|I-T^*T|\,\big|\,\mathfrak{D}_T = J_{T^*}(T^*)^{-1}J(I-T^*T)\,|\,\mathfrak{D}_T = \left(J_{T^*}(T^*)^{-1}J - T\right)\big|\,\mathfrak{D}_T.$$

Подставляя найденное значение оператора $J_{T^*}(T^*)^{-1}|I-T^*T|\big|\mathfrak{D}_T$ в формулу (1. 14), придем к (1. 13).

Впервые характеристическая функция в виде (1. 13) была определена Ю. Л. Шмульяном [10], который показал, что это определение эквивалентно данному М. С. Лившицем и В. П. Потаповым [9]. Последние, правда, рассматривали лишь случай $\mathfrak{d}_T = \mathfrak{d}_{T^*} < \infty$, где $\mathfrak{d}_T = \dim \overline{(I - T^* T)\mathfrak{H}}$. Одновременно Ю. Л. Шмульян показал, что определение этих двух авторов распространяется на общий случай любых $\mathfrak{d}_T, \mathfrak{d}_{T^*}$.

В случае сжатия, т. е. при $J = I$ функция (1. 13) принимает вид

$$(1.15)\qquad \Theta_{\mathscr{Y}}(\zeta) = \left(T - \zeta(I - TT^*)^{\frac{1}{2}}(I - \zeta T^*)^{-1}(I - T^*T)^{\frac{1}{2}}\right)\Big|\mathfrak{D}_T.$$

Именно этим равенством определяли характеристическую функцию Б. С.-Надь и Ч. Фояш, которые пришли к ней независимым и весьма оригинальным путем [13, 14].

Для обратимого оператора T и $\mathfrak{d}_T < \infty$ равенством (1. 11) в матричной форме определял характеристическую функцию А. В. Кужель [11].

§ 2. Основные тождества и неравенства

Тождества, получаемые в этом параграфе, обобщают равенство (1. 11); они играют важную роль в дальнейшем.

Теорема 2. 1. (Основные тождества) *Пусть* $\Theta_{\mathscr{Y}}(\zeta)$ — *характеристическая функция узла* (1. 5). *Тогда для любых* $\zeta, \eta \bar{\in} \sigma[(T^*)^{-1}]$ *справедливы соотношения*:

$$(2.1)\qquad \Theta^*_{\mathscr{Y}}(\eta) J \Theta_{\mathscr{Y}}(\zeta) = J - (1 - \zeta\bar{\eta}) R^*(I - \bar{\eta}T)^{-1}(I - \zeta T^*)^{-1} R,$$

$$(2.2)\qquad \Theta_{\mathscr{Y}}(\zeta) J \Theta^*_{\mathscr{Y}}(\eta) = J - (1 - \zeta\bar{\eta}) S^*(I - \zeta T^*)^{-1}(I - \bar{\eta}T)^{-1} S,$$

где $S = TR\Theta^{-1}_{\mathscr{Y}}(0)J$.

Доказательство. Согласно (1. 10)

$$\Theta^*_{\mathscr{Y}}(\eta) J \Theta_{\mathscr{Y}}(\zeta) = \left(J - F(\eta)\right)(K^*JK)^{-1}\left(J - F^*(\zeta)\right),$$

где $F(\zeta) = R^*(I - \bar{\zeta}T)^{-1}R$ и, стало быть,

$$\Theta^*_{\mathscr{Y}}(\eta) J \Theta_{\mathscr{Y}}(\zeta) = J(K^*JK)^{-1}J - F(\eta)(K^*JK)^{-1}J -$$
$$- J(K^*JK)^{-1}F^*(\zeta) + F(\eta)(K^*JK)^{-1}F^*(\zeta).$$

Преобразуем каждое из слагаемых правой части последнего равенства.

Так как

$$(2.3)\qquad R(K^*JK)^{-1}J = R(J - R^*R)^{-1}J = (I - RJR^*)^{-1}R = (T^*T)^{-1}R,$$

то

$$F(\eta)(K^*JK)^{-1}J = R^*T^{-1}(I - \bar{\eta}T)^{-1}(T^*)^{-1}R$$

и

$$J(K^*JK)^{-1}F^*(\zeta) = R^*T^{-1}(I - \zeta T^*)^{-1}(T^*)^{-1}R.$$

Формула $R(K^*JK)^{-1}R^* = T^{-1}(I-TT^*)(T^*)^{-1}$, вытекающая из (2. 3), непосредственно влечет за собой равенство

$$F(\eta)(K^*JK)^{-1}F^*(\zeta) = R^*T^{-1}(I-\bar{\eta}T)^{-1}(I-TT^*)(I-\zeta T^*)^{-1}(T^*)^{-1}R.$$

Таким образом,

$$\Theta_{\mathscr{Y}}^*(\eta)J\Theta_{\mathscr{Y}}(\zeta) = J(K^*JK)^{-1}J - R^*T^{-1}G(\eta,\zeta)(T^*)^{-1}R \tag{2. 4}$$

где

$$G(\eta,\zeta) = (I-\bar{\eta}T)^{-1}+(I-\zeta T^*)^{-1}-(I-\bar{\eta}T)^{-1}(I-TT^*)(I-\zeta T^*)^{-1}. \tag{2. 5}$$

Учитывая, что

$$(I-\bar{\eta}T)^{-1}TT^*(I-\zeta T^*)^{-1} = \frac{1}{\zeta\bar{\eta}}\big((I-\bar{\eta}T)^{-1}-I\big)\big((I-\zeta T^*)^{-1}-I\big),$$

получим

$$G(\eta,\zeta) = \left(1-\frac{1}{\zeta\bar{\eta}}\right)\big((I-\bar{\eta}T)^{-1}+(I-\zeta T^*)^{-1}-(I-\bar{\eta}T)^{-1}(I-\zeta T^*)^{-1}-I\big)+I,$$

или

$$G(\eta,\zeta) = I-\left(1-\frac{1}{\zeta\bar{\eta}}\right)\big((I-\bar{\eta}T)^{-1}-I\big)\big((I-\zeta T^*)^{-1}-I\big).$$

Отсюда следует

$$G(\eta,\zeta) = I+(1-\zeta\bar{\eta})T(I-\bar{\eta}T)^{-1}(I-\zeta T^*)^{-1}T^*. \tag{2. 6}$$

Подставляя выражение для $G(\eta,\zeta)$ из (2. 6) в (2. 4), получим

$$\Theta_{\mathscr{Y}}^*(\eta)J\Theta_{\mathscr{Y}}(\zeta) = J(K^*JK)^{-1}J - R^*T^{-1}(T^*)^{-1}R - \\ -(1-\zeta\bar{\eta})R^*(I-\bar{\eta}T)^{-1}(I-\zeta T^*)^{-1}R.$$

Наконец,

$$R^*(T^*T)^{-1}R = R^*(I-RJR^*)^{-1}R = J(J-R^*R)^{-1}R^*R = \\ = J(K^*JK)^{-1}R^*R = J(K^*JK)^{-1}J-J.$$

Из последних двух равенств вытекает тождество (2. 1).

Перейдем к доказательству тождества (2. 2). Для этого рассмотрим произведение

$$K^*J\Theta_{\mathscr{Y}}(\zeta)J\Theta_{\mathscr{Y}}^*(\eta)JK = \big(J-R^*(I-\zeta T^*)^{-1}R\big)J\big(J-R^*(I-\bar{\eta}T)^{-1}R\big).$$

Легко видеть, что

$$K^*J\Theta_{\mathscr{Y}}(\zeta)J\Theta_{\mathscr{Y}}^*(\eta)JK = J-R^*G_1(\eta,\zeta)R,$$

где

$$G_1(\eta,\zeta) = (I-\bar{\eta}T)^{-1}+(I-\zeta T^*)^{-1}-(I-\zeta T^*)^{-1}(I-T^*T)(I-\bar{\eta}T)^{-1}.$$

Аналогично тому, как было показано, что функция (2. 5) представима в виде (2. 6), доказывается равенство

$$G_1(\eta,\zeta) = I+(1-\zeta\bar{\eta})T^*(I-\zeta T^*)^{-1}(I-\bar{\eta}T)^{-1}T,$$

и поэтому

$$K^*\Theta_{\mathscr{Y}}(\zeta)J\Theta_{\mathscr{Y}}^*(\eta)JK = J-R^*R-(1-\zeta\bar{\eta})R^*T^*(I-\zeta T^*)^{-1}(I-\bar{\eta}T)^{-1}TR.$$

Умножая обе части последнего равенства слева и справа соответственно на $(K^*J)^{-1}$ и $(JK)^{-1}$, прийдем к тождеству (2. 2). Теорема доказана.

Следствие 2. 1. *Пусть $\Theta_{\mathscr{Y}}(\zeta)$ — характеристическая функция узла* (1. 5). *Тогда*

(2. 7) $$J-\Theta_{\mathscr{Y}}^*\left(\frac{1}{\bar{\zeta}}\right)J\Theta_{\mathscr{Y}}(\zeta)=0, \quad J-\Theta_{\mathscr{Y}}(\zeta)J\Theta_{\mathscr{Y}}^*\left(\frac{1}{\bar{\zeta}}\right)=0 \qquad (\zeta, \bar{\zeta}^{-1}\bar{\in}\sigma[(T^*)^{-}])$$

и

(2. 8) $$J-\Theta_{\mathscr{Y}}(\zeta)J\Theta_{\mathscr{Y}}^*(\zeta), \quad J-\Theta_{\mathscr{Y}}^*(\zeta)J\Theta_{\mathscr{Y}}(\zeta)\begin{cases}\geqq 0 & (|\zeta|<1)\\ =0 & (|\zeta|=1)\\ \leqq 0 & (|\zeta|>1)\end{cases} \quad (\zeta\bar{\in}\sigma[(T^*)^{-1}]).$$

Таким образом, $\Theta_{\mathscr{Y}}(\zeta)$ является двусторонним J-сжатием[6]), если $|\zeta|<1$ $(\zeta\bar{\in}\sigma[(T^*)^{-1}])$ и J-растяжением, если $|\zeta|>1$ $(\zeta\bar{\in}\sigma[(T^*)^{-1}])$.

Следствие 2. 2. *Если точка $\eta=e^{-i\varphi}$ $(0\leqq\varphi<2\pi)$ регулярна для оператора T, то при соответствующем выборе оператора K*

(2. 9) $$\Theta_{\mathscr{Y}}(\zeta) = I-(1-\zeta e^{i\varphi})JR^*(I-e^{i\varphi}T)^{-1}(I-\zeta T^*)^{-1}R.$$

В самом деле, из равенства (2. 8) вытекает, что оператор $\Theta^*(e^{-i\varphi})$ является J-унитарным. Следовательно, полагая в формуле (2. 1) $\eta=e^{-i\varphi}$ и учитывая, что Θ — функция $\Theta_{\mathscr{Y}}(\zeta)$ определяется с точностью до левого J-унитарного н ожителя, придем к равенству (2. 9).

§ 3. Характеристические функции дробно-линейных преобразований оператора

В этом параграфе исследуется связь между Θ-функциями оператора T и Θ-функциями его дробно линейных преобразований. Приводимые две теоремы отвечают случаям отображения круга в круг и круга в полуплоскость.

Теорема 3. 1. *Пусть $\mathscr{Y} = (\mathfrak{H}, \mathfrak{G};\ T,\ R,\ J)$ — некоторый узел. Если точка a $(|a|\neq 1)$ такова, что $a\bar{\in}\sigma(T)$ и $1/\bar{a}\bar{\in}\sigma(T)$, то совокупность $\mathscr{Y}_a$, состоящая из пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов $T_a = (T-aI)(I-\bar{a}T)^{-1}$, $R_a = (1-|a|^2)^{\frac{1}{2}}(I-aT^*)^{-1}R$, J образует узел и*

(3. 1) $$\Theta_{\mathscr{Y}}\left(\frac{\zeta+a}{1+\bar{a}\zeta}\right) = \Theta_{\mathscr{Y}_a}(\zeta).$$

[6]) Оператор $A\in[\mathfrak{H},\mathfrak{H}]$ называется J-сжатием (J-растяжением), если

$$J-A^*JA \geqq 0 \qquad (J-A^*JA\leqq 0)$$

и двусторонним J-сжатием (J-растяжением), если, кроме того,

$$J-AJA^* \geqq 0 \qquad (J-AJA^*\leqq 0).$$

Доказательство. Обратимость оператора T_a очевидна. Имеем

$$I - T_a^* T_a = I - (I - aT^*)^{-1}(T^* - \bar{a}I)(T - aI)(I - \bar{a}T)^{-1} =$$
$$= (I - aT^*)^{-1}[(I - aT^*)(I - \bar{a}T) - (T^* - \bar{a}I)(T - aI)](I - \bar{a}T)^{-1}.$$

Легко проверить, что выражение, стоящее в квадратных скобках, представимо в виде $(1 - |a|^2)(I - T^*T) = (1 - |a|^2)RJR^*$, и, поэтому,

$$I - T_a^* T_a = (1 - |a|^2)(I - aT^*)^{-1} RJR^*(I - \bar{a}T)^{-1} = R_a J R_a^*,$$

так что совокупность $\mathcal{Y}_a$ действительно образует узел.

На основании (2. 1)

$$\Theta_{\mathcal{Y}}^*(a) J \Theta_{\mathcal{Y}}(a) = J - (1 - |a|^2) R^*(I - \bar{a}T)^{-1}(I - aT^*)^{-1} R, \tag{3. 2}$$

$$\Theta_{\mathcal{Y}}^*(a) J \Theta_{\mathcal{Y}}(\zeta') = J - (1 - \bar{a}\zeta') R^*(I - \bar{a}T)^{-1}(I - \zeta' T^*)^{-1} R. \tag{3. 3}$$

Полагая $K_a = \Theta_{\mathcal{Y}}(a)$, получим из (3. 2) $K_a^* J K_a = J - R_a^* R_a$, и, следовательно,

$$\Theta_{\mathcal{Y}_a}(\zeta) = J\big(\Theta_{\mathcal{Y}}^*(a)\big)^{-1}\big(J - R_a^*(I - \zeta T_a^*)^{-1} R\big). \tag{3. 4}$$

Непосредственные вычисления показывают, что при $\zeta' = (\zeta + a)(1 + \bar{a}\zeta)^{-1}$

$$(1 - \bar{a}\zeta')(I - \zeta' T^*)^{-1} = (1 - |a|^2)(I - \zeta T_a^*)^{-1}(I - aT^*)^{-1}. \tag{3. 5}$$

Из (3. 3) вытекает уже равенство

$$\Theta_{\mathcal{Y}}(\zeta') = J\big(\Theta_{\mathcal{Y}}^*(a)\big)^{-1}\big(J - (1 - \bar{a}\zeta') R^*(I - \bar{a}T)^{-1}(I - \zeta' T^*)^{-1} R\big),$$

откуда с помощью (3. 5) и (3. 4) имеем:

$$\Theta_{\mathcal{Y}}(\zeta') = J\big(\Theta_{\mathcal{Y}}^*(a)\big)^{-1}\big(J - (1 - |a|^2) R^*(I - \bar{a}T)^{-1}(I - \zeta T_a^*)^{-1}(I - aT^*)^{-1} R\big) =$$
$$= J\big(\Theta_{\mathcal{Y}}^*(a)\big)^{-1}\big(J - R_a^*(I - \zeta T_a^*)^{-1} R_a\big) = \Theta_{\mathcal{Y}_a}(\zeta).$$

Теорема доказана.

Для случая сжатия и характеристической функции, определенной равенством (1. 15), эту теорему получили Б. С.-Надь и Ч. Фояш [14].

Произвольное дробно-линейное преобразование, отображающее круг на круг, может быть записано в виде

$$\zeta' = (\alpha\zeta + \beta)(\bar{\alpha} + \bar{\beta}\zeta)^{-1} \qquad (|\alpha|^2 > |\beta|^2) \tag{3. 6}$$

или $\zeta' = \varepsilon(\zeta + a)(1 + \bar{a}\zeta)^{-1}$, где $a = \beta\alpha^{-1}$, а $\varepsilon = \alpha/\bar{\alpha}$. Последнее отличается от преобразования, о котором идет речь в теореме 3. 1, лишь множителем ε. Отсюда уже легко заключить, что теорема 3. 1 распространяется на случай общего преобразования (3. 6), т. е. имеет место равенство

$$\Theta_{\mathcal{Y}}(\zeta') = \Theta_{\mathcal{Y}'}(\zeta),$$

где $\mathcal{Y}'$ — узел, состоящий из пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов

$$T' = (\bar{\alpha}T - \beta I)(\alpha I - \bar{\beta}T)^{-1}, \quad R' = (|\alpha|^2 - |\beta|^2)^{\frac{1}{2}}(\bar{\alpha}I - \beta T^*)^{-1} R, \; J.$$

Естественно, что при этом приходится требовать, чтобы точки $\alpha/\bar{\beta}$ и $\beta/\bar{\alpha}$ не принадлежали спектру оператора T. Заметим, что переходы от оператора T к T' и от переменной ζ к ζ' совершаются с помощью взаимно обратных преобразований.

Теорема 3. 2. *Пусть* $\mathscr{Y} = (\mathfrak{H}, \mathfrak{G};\ T, R, J)$ — *некоторый узел. Если точка* $\eta = e^{-i\varphi}$ *регулярна для оператора* T, *то совокупность, состоящая из пространств* $\mathfrak{H}$, $\mathfrak{G}$ *и операторов*

$$A = i(I+e^{i\varphi}T)(I-e^{i\varphi}T)^{-1}, \quad S = (I-e^{-i\varphi}T^*)^{-1}R, \quad J \tag{3.7}$$

образует $\mathscr{W}$-*узел и*

$$\mathscr{W}\left(i\frac{\zeta e^{i\varphi}+1}{\zeta e^{i\varphi}-1}\right) = \Theta_{\mathscr{Y}}(\zeta). \tag{3.8}$$

Доказательство. Легко показать, что

$$A - A^* = i(I-e^{-i\varphi}T^*)^{-1}[(I-e^{-i\varphi}T^*)(I+e^{i\varphi}T)+$$
$$+(I+e^{-i\varphi}T^*)(I-e^{i\varphi}T)](I-e^{i\varphi}T)^{-1}$$

и

$$(I-e^{-i\varphi}T^*)(I+e^{i\varphi}T)+(I+e^{-i\varphi}T^*)(I-e^{i\varphi}T) = 2(I-T^*T).$$

Следовательно,

$$\operatorname{Im} A = (I-e^{-i\varphi}T^*)^{-1}(I-T^*T)(I-e^{i\varphi}T)^{-1} = SJS^*.$$

Отсюда вытекает, что совокупность пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов A, S, J действительно образует $\mathscr{W}$-узел.

Пусть $\zeta' = i(\zeta e^{i\varphi}+1)(1-e^{i\varphi}\zeta)^{-1}$. Так как

$$A^*-\zeta' I = -i(I-e^{-i\varphi}T^*)^{-1}(I+e^{i\varphi}T^*)-i(\zeta e^{i\varphi}+1)(1-\zeta e^{i\varphi})^{-1}I =$$
$$= -i(1-\zeta e^{i\varphi})^{-1}(I-e^{-i\varphi}T^*)^{-1}[(1-\zeta e^{i\varphi})(I+e^{-i\varphi}T^*)+(\zeta e^{i\varphi}+1)(I-e^{-i\varphi}T^*)]$$

и

$$(1-\zeta e^{i\varphi})(I+e^{-i\varphi}T^*)+(\zeta e^{i\varphi}+1)(I-e^{-i\varphi}T^*) = 2(I-T^*T),$$

то

$$(A^*-\zeta' I)^{-1} = -\frac{1-\zeta e^{i\varphi}}{2i}(I-\zeta T^*)^{-1}(I-e^{-i\varphi}T^*)^{-1}. \tag{3.9}$$

Как известно

$$W(\zeta') = I+2iJS^*(A^*-\zeta' I)^{-1}S.$$

Ввиду (3. 7) и (3. 9)

$$W(\zeta') = I-(1-\zeta e^{i\varphi})JR^*(I-e^{i\varphi})^{-1}(I-\zeta T^*)^{-1}R,$$

что в сочетании со следствием 2. 2 дает утверждение теоремы.

К этой теореме можно сделать замечание, аналогичное тому, которое было сделано к предыдущей теореме. Преобразование $\zeta' = i(\zeta e^{i\varphi}+I)(1-\zeta e^{i\varphi})^{-1}$ можно заменить любым дробно-линейным преобразованием, отображающим круг на верхнюю полуплоскость, лишь бы оператор T допускал обратное преобразование. Это утверждение вытекает из доказанной теоремы 3. 2 и следующего предложения.

Теорема 3. 3. *Пусть* $\mathscr{W} = (\mathfrak{H}, \mathfrak{G};\ A, S, J)$ — *некоторый узел, а* $\zeta' = (\alpha\zeta+\beta)(\gamma\zeta+\delta)^{-1}$ (α, β, γ, δ — *вещественные числа и* $\alpha\delta-\gamma\beta>0$) — *какое-либо преобразование, отображающее верхнюю полуплоскость на себя. Если оператор T допускает обратное преобразование*

$$A' = (\beta I - \delta A)(\gamma A - \alpha I)^{-1} \qquad (\alpha/\gamma \overline{\in} \sigma(A)),$$

то

$$SW(\zeta') = W'(\zeta), \tag{3.10}$$

где $W'(\zeta)$ — характеристическая функция узла $\mathscr{W}'$, состоящего из пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов

$$A' \quad S' = (\alpha\delta-\beta\gamma)^{\frac{1}{2}}(\gamma A - \alpha I)^{-1} S, \quad J,$$

а U — J-унитарный оператор, вычисляемый по формуле $U = U^*\left(\dfrac{\alpha}{\gamma}\right)J$.

Доказательство: аналогично доказательству теоремы 3. 2. При этом вместо формулы (2. 1) используется следующая (см. [4])

$$W^*(\eta)JW(\zeta) = J + 2i(\bar\eta - \zeta)S^*(A-\bar\eta I)^{-1}(A^*-\zeta I)^{-1}S.$$

§ 4. Унитарно-эквивалентные и простые узлы

Узлы

$$\mathscr{Y}_1 = (\mathfrak{H}_1, \mathfrak{G}; T_1, R_1, J), \quad \mathscr{Y}_2 = (\mathfrak{H}_2, \mathfrak{G}; T_2, R_2, J) \tag{4.1}$$

назовем *унитарно-эквивалентными*[7]), если существует изометрический оператор $U \in [\mathfrak{H}_1, \mathfrak{H}_2]$ такой, что

$$UT_1 = T_2 U, \quad UR_1 = R_2. \tag{4.2}$$

Если узлы (4. 1) унитарно эквивалентны, то, как легко видеть, $\Theta_{\mathscr{Y}_1}(\zeta) = \Theta_{\mathscr{Y}_2}(\zeta)$. Обратное утверждение, вообще говоря, неверно. Действительно, если узел $\mathscr{Y}_2$ является *несущественным расширением* узла $\mathscr{Y}_1$, т. е.

$$\mathfrak{H}_2 = \mathfrak{H}_1 \oplus \mathfrak{H}_0, \qquad T_1 f = T_2 f \quad (f \in \mathfrak{H}_1), \qquad R_1 g = R_2 g \quad (g \in \mathfrak{G})$$

и сужение T_2 на пространство $\mathfrak{H}_0$ есть унитарный оператор, то $\Theta_{\mathscr{Y}_1}(\zeta) = \Theta_{\mathscr{Y}_2}(\zeta)$, хотя равенство (4. 2) и не выполняется.

---

[7]) Можно было бы ввести понятие унитарной эквивалентности без предположения, что узлы содержат одни и те же пространства $\mathfrak{G}$, однако для дальнейшего это не существенно.

Таким образом, обратное утверждение может иметь место лишь при некотором дополнительном условии. Для того, чтобы его сформулировать, положим для узла $\mathscr{Y} = (\mathfrak{H}, \mathfrak{G}; T, R, J)$:

$$\mathfrak{H}_{\mathscr{Y}} = \bigvee_{n=-\infty}^{\infty} T^n R\mathfrak{G}. \text{ [8]}$$

Очевидно, $T\mathfrak{H}_{\mathscr{Y}} \subset \mathfrak{H}_{\mathscr{Y}}$. Так как $\mathfrak{H}_{\mathscr{Y}} \supset R\mathfrak{G}$, то $\mathfrak{H}_{\mathscr{Y}} \supset RJR^*\mathfrak{H} = (J - T^*T)\mathfrak{H}$, и, стало быть $(I - T^*T)\mathfrak{H}_{\mathscr{Y}}^{\perp} = 0$ ($\mathfrak{H}_{\mathscr{Y}}^{\perp} = \mathfrak{H} \ominus \mathfrak{H}_{\mathscr{Y}}$). С другой стороны, из равенств

$$T^*T^n R\mathfrak{G} = -(I - T^*T)T^{n-1}R\mathfrak{G} + T^{n-1}R\mathfrak{G} = -RJR^*T^{n-1}R\mathfrak{G} + T^{n-1}R\mathfrak{G}$$

следует, что $T^*\mathfrak{H}_{\mathscr{Y}} \subset \mathfrak{H}_{\mathscr{Y}}$. Поэтому оператор T индуцирует в $\mathfrak{H}_{\mathscr{Y}}^{\perp}$ унитарный оператор.

Узел $\mathscr{Y}$ назовем *простым*, если $\mathfrak{H}_{\mathscr{Y}} = \mathfrak{H}$.

В силу инвариантности пространства $\mathfrak{H}_{\mathscr{Y}}$ относительно операторов T и T^* пространство

$$\mathfrak{H}_{\mathscr{Y}^*} = \bigvee_{n=-\infty}^{\infty} (T^*)^n R\mathfrak{G}$$

содержится в $\mathfrak{H}_{\mathscr{Y}}$. Пользуясь формулами

$$\begin{aligned} T^n R\mathfrak{G} &= -(T^*)^{-1}(I - T^*T)T^{n-1}R\mathfrak{G} + (T^*)^{-1}T^{n-1}R\mathfrak{G} = \\ &= -(T^*)^{-1}RJR^*T^{n-1}R\mathfrak{G} + (T^*)^{-1}T^{n-1}R\mathfrak{G}, \end{aligned}$$

$$T^{-n}R\mathfrak{G} = (I - T^*T)T^{-n}R\mathfrak{G} + T^*T^{-n+1}R\mathfrak{G} = RJR^*T^{-n}R\mathfrak{G} + T^*T^{-n+1}R\mathfrak{G},$$

легко доказать, что $\mathfrak{H}_{\mathscr{Y}} \subset \mathfrak{H}_{\mathscr{Y}^*}$. Следовательно $\mathfrak{H}_{\mathscr{Y}} = \mathfrak{H}_{\mathscr{Y}^*}$. С помощью этого равенства, используя хорошо известные приемы (см. [19, 12] и [4]), доказывается требуемое «обратное» утверждение.

Теорема 4. 1.[9]) *Если узлы* (4. 1) *простые, и в некоторой окрестности точки* $\zeta = 0$ *имеет место тождество*

$$\Theta_{\mathscr{Y}_1}(\zeta) \equiv \Theta_{\mathscr{Y}_2}(\zeta), \tag{4. 3}$$

то узлы $\mathscr{Y}_1$ *и* $\mathscr{Y}_2$ *унитарно эквивалентны.*

Доказательство. В самом деле, из указанного тождества и (2. 1) следует, что

$$R_1^*(I - \bar{\eta}T_1)^{-1}(I - \zeta T_1^*)^{-1}R_1 = R_2^*(I - \bar{\eta}T_2)^{-1}(I - \zeta T_2^*)^{-1}R_2,$$

откуда

$$R_1^*T_1^n(T_1^*)^n R_1 = R_2^*T_2^n(T_2^*)^n R_2 \qquad (n = 0, 1, 2, \ldots). \tag{4. 4}$$

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[8]) Символом $\bigvee_{n \in N} \mathfrak{G}_n$, где $\mathfrak{G}_n$ — подмножества из $\mathfrak{H}$, обозначается, как в [14], линейная замкнутая оболочка объединения всех $\mathfrak{G}_n (n \in N)$.

[9]) Родственная теорема, но при других условиях относительно оператора T имеется в работе [26].

В силу отношений (2. 7) тождество (4. 3) имеет место и при достаточно больших ζ, т.е. в окрестности точки $\zeta=0$ также

$$\Theta_{\mathscr{Y}_1}\left(\frac{1}{\zeta}\right) \equiv \Theta_{\mathscr{Y}_2}\left(\frac{1}{\zeta}\right). \tag{4. 5}$$

Следовательно,

$$R_1^*(\bar{\eta}I-T_1)^{-1}(\zeta I-T_1^*)^{-1}R_1 = R_2(\bar{\eta}I-T_2)^{-1}(\zeta I-T_2^*)^{-1}R_2$$

и, значит, соотношения (4. 4) имеют место также при $n=-1, -2, \ldots$.

Таким образом, оператор U, определенный равенством

$$U(T_1^*)^n R_1 = (T_2^*)^n R_2 \qquad (n=0, \pm 1, \pm 2, \ldots) \tag{4. 6}$$

изометричен. Он действует из $\mathfrak{U}_1$ в $\mathfrak{U}_2$, где

$$\mathfrak{U}_j = \bigvee_{n=-\infty}^{\infty} (T_j^*)^n R_j \mathfrak{G} \qquad (j=1, 2).$$

В силу простоты узлов $\mathscr{Y}_j$ множества $\mathfrak{U}_j$ плотны в $\mathfrak{H}_j$ и, поэтому, оператор U может быть расширен до изометрического оператора, действующего из $\mathfrak{H}_1$ в $\mathfrak{H}_2$. Обозначим это расширение тем же символом U.

На основании (4. 6) $UR_1=R_2$ и

$$UT_1^*(T_1^*)^n R_1 = T_2^*(T_2^*)^n R_2 = T_2^* U(T_1^*)^n R_1.$$

Отсюда вытекает, что $UT_1^* = T_2^* U$ и, стало быть, $UT_1 = T_2 U$.

§ 5. Основная лемма

При восстановлении $\mathscr{Y}$-узла по его характеристической функции, фундаментальную роль играет следующее предложение:

Лемма 5. 1. *Пусть $\mathfrak{H}$ и $\mathfrak{G}$ — некоторые гильбертовы пространства, $J(\in[\mathfrak{G}, \mathfrak{G}])$ — сигнатурный оператор $(J^2=I,\ J^*=J)$ и $\omega(\zeta)$ — оператор-функция, определенная равенством*

$$\omega(\zeta) = S^*(U+\zeta I)(U-\zeta J)^{-1}S, \tag{5. 1}$$

где U — унитарный оператор, действующий в $\mathfrak{H}$, а S — оператор из множества $[\mathfrak{G}, \mathfrak{H}]$.

Если операторы $I-JS^*S$, $I+JS^*S$ обратимы, то в достаточно малой окрестности $\mathfrak{U}$ точки $\zeta=0$ оператор-функция $I+J\omega(\zeta)$ обратима, а оператор-функция

$$\vartheta(\zeta) = \big(I-J\omega(\zeta)\big)\big(I+J\omega(\zeta)\big)^{-1} \qquad (\zeta\in\mathfrak{U}) \tag{5. 2}$$

допускает представление

$$\vartheta(\zeta) = J(\vartheta^*(0))^{-1}(J - R^*(I-\zeta T^*)^{-1}R), \tag{5.3}$$

где

$$R = 2(I+SJS^*)^{-1}S, \quad T = U(I+SJS^*)^{-1}(I-SJS^*). \tag{5.4}$$

Доказательство. Из обратимости оператора $I+J\omega(0) = I+JS^*S$ вытекает обратимость операторов

$$I+J\omega(\zeta) = I+JS^*(U+\zeta I)(U-\zeta I)^{-1}S \qquad (\zeta\in\mathfrak{U}),$$
$$I+SJS^*(U+\zeta I)(U-\zeta I)^{-1} \quad (\zeta\in\mathfrak{U}), \ I+SJS^*.$$

Кроме того, справедливы равенства

$$\big(I+SJS^*(U+\zeta I)(U-\zeta I)^{-1}\big)^{-1}S = S\big(I+J\omega(\zeta)\big)^{-1} \qquad (\zeta\in\mathfrak{U}) \tag{5.5}$$

и

$$(I+JS^*S)^{-1}JS^* = JS^*(I+SJS^*)^{-1}. \tag{5.6}$$

Так как

$$\vartheta(\zeta) = 2\big(I+J\omega(\zeta)\big)^{-1}-I, \quad \vartheta(0) = 2(I+JS^*S)^{-1}-I,$$

то разность $\vartheta(\zeta)-\vartheta(0)$ можно представить в виде

$$\vartheta(\zeta)-\vartheta(0) = 2\big(I+J\omega(\zeta)\big)^{-1}-2(I+JS^*S)^{-1} =$$
$$= 2(I+JS^*S)^{-1}J\big(S^*S-\omega(\zeta)\big)\big(I+J\omega(\zeta)\big)^{-1}.$$

Учитывая, что

$$S^*S-\omega(\zeta) = -2\zeta S^*(U-\zeta I)^{-1}S,$$

и пользуясь (5.5) и (5.6), получим

$$\vartheta(\zeta)-\vartheta(0) = -4\zeta JS^*(I+SJS^*)^{-1}(U-\zeta I)^{-1}\big(I+SJS^*(U+\zeta I)(U-\zeta I)^{-1}\big)^{-1}S.$$

С помощью несложных преобразований легко установить соотношение

$$(U-\zeta I)^{-1}\big(I+SJS^*(U+\zeta I)(U-\zeta I)^{-1}\big)^{-1}S = U^*(I-\zeta T^*)^{-1}R,$$

где операторы T и R определены формулами (5.4) и, стало быть,

$$\vartheta(\zeta) = \vartheta(0)+A(\zeta),$$

где

$$A(\zeta) = -2\zeta JS^*(I+SJS^*)^{-1}\mathcal{U}^*(I-\zeta T^*)^{-1}R$$

Из обратимости $I-JS^*S$ вытекает обратимость $\vartheta(0)$ и поэтому

$$\vartheta(\zeta) = J(\vartheta^*(0))^{-1}(\vartheta^*(0)J\vartheta(0)+\vartheta^*(0)JA^*(\zeta)). \tag{5.7}$$

Рассмотрим в отдельности каждое из слагаемых, стоящих в квадратных скобках.

Исходя из равенства

$$\vartheta^*(0)J\vartheta(0) = (I+S^*SJ)^{-1}(I-S^*SJ)J(I-JS^*S)(I+JS^*S)^{-1},$$

получим

$$\vartheta^*(0)J\vartheta(0)-J=$$
$$=(I+S^*SJ)^{-1}[(I-S^*SJ)J(I-JS^*S)-(I+S^*SJ)J(I+JS^*S)](I+JS^*S)^{-1}=$$
$$=-4(I+S^*SJ)^{-1}S^*S(I+JS^*S)^{-1}.$$

Таким образом,

$$\vartheta^*(0)J\vartheta(0)=J-R^*R. \tag{5.8}$$

Так как $\vartheta^*(0)S^*=S^*(I+SJS^*)^{-1}(I-SJS^*)$, то

$$\vartheta^*(0)JA(\zeta)= \tag{5.9}$$
$$=-2\zeta S^*(I+SJS^*)^{-1}(I-SJS^*)(I+SJS^*)^{-1}U^*(I-\zeta T^*)^{-1}R=$$
$$=-\zeta R^*T^*(I-\zeta T^*)^{-1}R=R^*R-R^*(I-\zeta T^*)^{-1}R.$$

Из (5. 7)—(5. 9) следует, что

$$\vartheta(\zeta)=J(\vartheta^*(0))^{-1}(J-R^*(I-\zeta T^*)^{-1}R).$$

Лемма доказана.

§ 6. Восстановление узла по его характеристической функции

1. Пусть $\mathfrak{G}$ — некоторое гильбертово пространство, а $J(\in[\mathfrak{G},\mathfrak{G}])$ — *сигнатурный оператор*, т. е. $J^2=I$, $J^*=J$. Обозначим через $\mathfrak{A}(J)$ множество всех оператор-функций $\Theta(\zeta)$ со значениями из $[\mathfrak{G},\mathfrak{G}]$, удовлетворяющих условиям:

I. функция $\Theta(\zeta)$ определена и голоморфна в некоторой окрестности точки $\zeta=0$;

II. оператор $\Theta(0)$ является обратимым двусторонним J-сжатием $(\Theta^*(0)J\Theta(0)\leqq J,\ \Theta(0)J\Theta^*(0)=J)$;

III. оператор-функция

$$\Omega(\zeta)=J(I-U_0^{-1}\Theta(\zeta))(I+U_0^{-1}\Theta(\zeta))^{-1}, \tag{6.1}$$

где оператор U_0 взят из J-полярного представления оператора $\Theta(0)$[10]),

$$\Theta(0)=U_0H_0 \qquad (U_0^*JU_0=U_0JU_0=J,\ (JH_0)^*=JH_0,\ \sigma(H_0)\subset(0,\infty)), \tag{6.2}$$

допускает аналитическое продолжение на весь круг $|\zeta|<1$ и это продолжение удовлетворяет неравенству

$$\Omega(\zeta)+\Omega^*(\zeta)\gg 0 \qquad (|\zeta|<1). \tag{6.3}$$

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[10]) По теореме Потапова—Гинзбурга [23, 24] (см. также [25]) каждое обратимое двустороннее J-сжатие допускает единственное J-полярное представление вида (6.2). Так как оператор $I+U_0^{-1}\Theta(0)$ $(=I+H_0)$ обратим, то оператор $I+U_0^{-1}\Theta(\zeta)$ обратим в некоторой окрестности точки $\zeta=0$. В этой окрестности задана функция (6.1).

Теорема 6.1. *Пусть голоморфная в некоторой окрестности* $\mathfrak{U}$ *точки* $\zeta=0$ *оператор-функция* $\Theta(\zeta)$ *принимает значения из* $[\mathfrak{G}, \mathfrak{G}]$.

Для того, чтобы существовал узел $\mathcal{Y} = (\mathfrak{H}, \mathfrak{G};\ T,\ R,\ J)$ *такой, что* $\Theta(\zeta) = \Theta_{\mathcal{Y}}(\zeta)(\zeta\in\mathfrak{U})$, *необходимо и достаточно чтобы* $\Theta(\zeta)\in\mathfrak{A}(J)$.

При выполнении этого условия узел $\mathcal{Y}$ *может быть выбран простым, и тогда он будет определен с точностью до унитарной эквивалентности.*

Доказательство. Пусть существует узел $\mathcal{Y} = (\mathfrak{H}, \mathfrak{G};\ T,\ R,\ J)$, такой, что $\Theta(\zeta)=\Theta_{\mathcal{Y}}(\zeta)$ и $\Theta_{\mathcal{Y}}(0) = U_0H_0$ — J-полярное представление оператора $\Theta_{\mathcal{Y}}(0)$ вида (6. 2). Очевидно, что функция $\Theta_{\mathcal{Y}}(\zeta)$ удовлетворяет условиям I, II, сформулированным при определении класса $\mathfrak{A}(J)$. Покажем, что выполняется также и условие III.

Пользуясь равенством $\Theta_{\mathcal{Y}}^*(0)J\Theta_{\mathcal{Y}}(0) = J-R^*R$ и леммой 1. 1, получим

(6. 4) $$H_0 = (I-JR^*R)^{\frac{1}{2}}.$$

Рассмотрим обычное полярное представление $T=UH$ оператора T и положим

(6. 5) $$S = (I+H)^{-1}R.$$

Легко видеть, что

(6. 6) $$H = (T^*T)^{\frac{1}{2}} = (I-RJR^*)^{\frac{1}{2}}$$

и, кроме того,

$$JR^*\big(I+(I-RJR^*)^{\frac{1}{2}}\big)^{-1} = \big(I+(I-JR^*R)^{\frac{1}{2}}\big)^{-1}JR^*,$$

$$\big(I+(I-RJR^*)^{\frac{1}{2}}\big)^{-1}R = R\big(I+(I-JR^*R)^{\frac{1}{2}}\big)^{-1}.$$

Следовательно,

$$JR^*(I+H)^{-1} = (I+H_0)^{-1}JR^*,\quad (I+H)^{-1}R = R(I+H_0)^{-1}$$

или, ввиду (6. 5)

(6. 7) $$JS^* = (I+H_0)^{-1}JR^*,\quad S = R(I+H_0)^{-1}.$$

Отсюда имеем

$$JS^*S = (I+H_0)^{-1}JR^*R(I+H_0)^{-1},$$

и поэтому

$$I+JS^*S = (I+H_0)^{-1}[(I+H_0)^2+JR^*R](I+H_0)^{-1}.$$

На основании (6. 4)

$$(I+H_0)^2+JR^*R = I+2H_0+H_0^2+JR^*R =$$

$$= 2I+2H_0+H_0^2-(I-JR^*R) = 2(I+H_0)$$

и, значит,

(6. 8) $$I+JS^*S = 2(I+H_0)^{-1}.$$

Аналогичным образом доказывается равенство

(6. 9) $$I-JS^*S = 2H_0(I+H_0)^{-1}.$$

В силу (6. 8) и (6. 9)

$$H_0 = (I - JS^*S)(I + JS^*S)^{-1}. \tag{6.10}$$

Пользуясь опять равенством (6. 5), получим

$$SJS^* = (I+H)^{-1}RJR^*(I+H)^{-1},$$

откуда вытекает

$$I + SJS^* = (I+H)^{-1}\left((I+H)^2 + RJR^*\right)(I+H)^{-1} \tag{6.11}$$

и

$$I - SJS^* = (I+H)^{-1}\left((I+H)^2 - RJR^*\right)(I+H)^{-1}. \tag{6.12}$$

Так как

$$(I+H)^2 + RJR^* = I + 2H + H^2 + I - T^*T = 2(I+H),$$

$$(I+H)^2 - RJR^* = I + 2H + H^2 - I - T^*T = 2H(I+H),$$

то подставляя найденные здесь значения в формулы (6. 11) и (6. 12), получим

$$I + SJS^* = 2(I+H)^{-1}, \quad I - SJS^* = 2H(I+H)^{-1}.$$

Отсюда далее следует

$$H = (I - SJS^*)(I + SJS^*)^{-1} \tag{6.13}$$

и

$$(I+H)S = 2(I+SJS^*)^{-1}S. \tag{6.14}$$

Пользуясь полярным представлением $T = UH$ оператора T и (6. 13), получаем

$$T = U(I - SJS^*)(I + SJS^*)^{-1}. \tag{6.15}$$

Из (6. 14) и (6. 15) следует

$$R = 2(I + SJS^*)^{-1}S. \tag{6.16}$$

Оператор $\Theta_{\mathscr{Y}}(0) = U_0H_0$ обратим. Равенства (6. 8) и (6. 9) показывают, что обратимы операторы $I + JS^*S$, $I - JS^*S$ и таким образом применима лемма 5. 1. Из нее и равенств (6. 15), (6. 16) вытекает, что дробно-линейное преобразование функции

$$\omega(\zeta) = S^*(U + \zeta I)(U - \zeta I)^{-1}S, \tag{6.17}$$

определенное в некоторой окрестности $\mathfrak{U}$ точки $\zeta = 0$ равенством

$$\vartheta(\zeta) = \left(I - J\omega(\zeta)\right)\left(I + J\omega(\zeta)\right)^{-1}, \tag{6.18}$$

допускает представление

$$\vartheta(\zeta) = J\left(\vartheta^*(0)\right)^{-1}\left(J - R^*(I - \zeta T^*)^{-1}R\right). \tag{6.19}$$

В силу (6. 17), (6. 18) $\vartheta(0) = (I - JS^*S)(I + JS^*S)^{-1}$, а это, ввиду (6. 10), означает, что $\vartheta(0) = H_0 = U_0^{-1}\Theta_{\mathscr{Y}}(0)$, и поэтому

$$\left(\vartheta^*(0)\right)^{-1} = \left(\Theta_{\mathscr{Y}}^*(0)(U_0^*)^{-1}\right)^{-1} = U_0^*\left(\Theta_{\mathscr{Y}}^*(0)\right)^{-1} = JU_0^{-1}J\left(\Theta_{\mathscr{Y}}^*(0)\right)^{-1}. \tag{6.20}$$

Из (6. 19) и (6. 20) вытекает равенство

$$U_0\vartheta(\zeta) = J\big(\Theta_{\mathcal{Y}}^*(0)\big)^{-1}\big(J-R^*(I-\zeta T^*)^{-1}R\big) = \Theta_{\mathcal{Y}}(\zeta),$$

откуда с помощью (6. 18) находим

$$U_0^{-1}\Theta_{\mathcal{Y}}(\zeta) = \big(I-J\omega(\zeta)\big)\big(I+J\omega(\zeta)\big)^{-1} \qquad (\zeta\in\mathfrak{U}).$$

Следовательно,

$$\omega(\zeta) = J\big(I-U_0^{-1}\Theta_{\mathcal{Y}}(\zeta)\big)\big(I+U_0^{-1}\Theta_{\mathcal{Y}}(\zeta)\big)^{-1} \qquad (\zeta\in\mathfrak{U}).$$

Эта формула показывает, что функция $\omega(\zeta)$ является расширением на весь круг $|\zeta|<1$ оператор-функции $\Omega(\zeta)$, определенной равенством (6. 1). Неравенство (6. 3) легко проверяется с помощью непосредственных вычислений. Таким образом, условие III выполняется и $\Theta_{\mathcal{Y}}(\zeta)\in\mathfrak{A}(J)$.

Покажем теперь достаточность условий теоремы. Пусть $\Theta(\zeta)\in\mathfrak{A}(J)$. Рассмотрим оператор-функцию $\Omega(\zeta)$, определенную равенством (6. 1). Заметим, что

(6.21) $$\Omega(0) = J(I-H_0)(I+H_0)^{-1},$$

и, поэтому,

$$\Omega^*(0) = (I-H_0^*)(I+H_0^*)^{-1}J = J(I-JH_0^*J)(I+JH_0^*J) = \Omega(0).$$

Отсюда и из условия III (см. также [4], стр. 36) вытекает существование неубывающей неотрицательной оператор-функции $F(t)$ $(0\leq t\leq 2\pi)$ такой, что

$$\Omega(\zeta) = \int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\,dF(t)$$

В силу обобщенной теоремы Наймарка [4], найдется пространство $\mathfrak{H}(\supset\mathfrak{G})$, разложение единицы $E(t)$, действующее в $\mathfrak{H}$, и оператор $S\in[\mathfrak{G},\mathfrak{H}]$, такие, что $F(t) = S^*E(t)S$ $(0\leq t\leq 2\pi)$. Следовательно,

(6.22) $$\Omega(\zeta) = S^*\int_0^{2\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta}\,dE(t)\mathscr{S} = \mathscr{S}^*(U+\zeta I)(U-\zeta I)^{-1}S,$$

где $U = \int_0^{2\pi} e^{it}\,dE(t)$.

Будем предполагать, что пространство $\mathfrak{H}$ минимально, т. е. $S\overline{\mathfrak{G}}=\mathfrak{H}$.

На основании (6. 22) $\Omega(0) = S^*S$. Пользуясь (6. 21), находим, что $I+JS^*S = 2(I+H_0)^{-1}$. Отсюда следует обратимость оператора $I+JS^*S$ и равенство $(I-JS^*S)(I+JS^*S)^{-1} = H_0$.

Так как оператор H_0 обратим, то обратимым является также оператор $I-JS^*S$ и, значит, выполняются все требования леммы 5. 1. Применяя равенство (6. 1), нетрудно доказать соотношение

$$S_0^{-1}\Theta(\zeta) = \big(I-J\Omega(\zeta)\big)\big(I+J\Omega(\zeta)\big)^{-1},$$

которое в сочетании с леммой 5.1 дает

$$U_0^{-1}\Theta(\zeta) = J\big((U_0^{-1}\Theta(0))^*\big)^{-1}\big(J-R^*(I-\zeta T^*)^{-1}R\big), \tag{6.23}$$

где $\qquad R = 2(I+SJS^*)^{-1}S, \quad T = U(I+SJS^*)^{-1}(I-SJS^*).$

Рассмотрим произведение

$$T^*T = (I+SJS^*)^{-1}(I-SJS^*)^2(I+SJS^*)^{-1}.$$

Так как

$$I-T^*T = (I+SJS^*)^{-1}[(I+SJS^*)^2-(I-SJS^*)^2](I+SJS^*)^{-1},$$

и $(I+SJS^*)^2-(I-SJS^*)^2 = 4SJS^*$, то

$$I-T^*T = 4(I+SJS^*)^{-1}SJS^*(I+SJS^*)^{-1} = RJR^*$$

и, следовательно, совокупность, состоящая из пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов T, R, J, образует $\mathcal{Y}$-узел. Этот узел простой, т. к. $\overline{R\mathfrak{G}} = \mathfrak{H}$. Поскольку

$$\big((U_0^{-1}\Theta(0))^*\big)^{-1} = U_0^*\big(\Theta^*(0)\big)^{-1} = JU_0^{-1}J\big(\Theta^*(0)\big)^{-1},$$

то равенство (6.23) можно преобразовать к виду

$$\Theta(\zeta) = J\big(\Theta^*(0)\big)^{-1}\big(J-R^*(I-\zeta T^*)^{-1}R\big),$$

который показывает, что $\Theta(\zeta) = \Theta_{\mathcal{Y}}(\zeta)$. Теорема доказана.

2. Обозначим через $\mathfrak{U}_\infty(\mathfrak{H})$ множество всех обратимых операторов $T\in[\mathfrak{H}, \mathfrak{H}]$, для которых $I-T^*T\in\mathfrak{S}_\infty$.[11]) Легко проверяется, что вместе с T также и $T^*\in\mathfrak{U}_\infty(\mathfrak{H})$.

Для Θ-функций операторов $T\in\mathfrak{U}_\infty(\mathfrak{H})$ может быть получена более простая аналитическая характеристика, нежели та, которая дается теоремой 5.1.

С этой целью введем в рассмотрение класс аналитических функций $\mathfrak{A}_\infty(J)$, где J — сигнатурный оператор из $[\mathfrak{G}, \mathfrak{G}]$. Условимся писать, что функция $\Theta(\zeta)$ со значениями из $[\mathfrak{G}, \mathfrak{G}]$ принадлежит классу $\mathfrak{A}_\infty(J)$, если она обладает следующими свойствами:

а) функция $\Theta(\zeta)$ голоморфна внутри единичного круга, за исключением не более, чем счетного множества точек ε_Θ, причем $0\bar{\in}\varepsilon_\Theta$,

б) оператор $\Theta(0)$ обратим, а оператор $J-\Theta^*(0)J\Theta(0)$ вполне непрерывен;

в) для любой своей точки ζ голоморфности $\Theta(\zeta)$ является двусторонним J-нерастягивающим оператором.

Этот класс функций (даже при несколько более общих предположениях) детально изучался Ю. П. Гинзбургом [27]. В частности, этот автор показал, что каждая функция $\Theta(\zeta)\in\mathfrak{A}_\infty(J)$ допускает представление

$$\Theta(\zeta) = U_0\Theta_0(\zeta), \tag{6.24}$$

[11]) Символом $\mathfrak{S}_\infty$ обозначают симметрично нормированный (с.-н.) идеал, состоящий из всех вполне непрерывных операторов $T\in[\mathfrak{H}, \mathfrak{H}]$ (см. [28]).

где U_0 — постоянный J-унитарный оператор, взятый из J-полярного представления оператора $\Theta(0)$ вида (6. 2), а $\Theta_0(\zeta)-I$ — вполне непрерывный оператор во всех точках голоморфности $\Theta(\zeta)$. Кроме того, Ю. П. Гинзбургом было показано, что множество ε_Θ состоит лишь из изолированных точек.

Легко видеть, что Θ-функция любого оператора $T\in\mathfrak{U}_\infty(\mathfrak{H})$ принадлежит классу $\mathfrak{A}_\infty(J)$. Оказывается, верно и обратное утверждение:

Теорема 6. 2. *Если* $\Theta(\zeta)\in\mathfrak{A}_\infty(J)$, *то существует простой узел* $Y=(\mathfrak{H},\mathfrak{G};\mathrm{T},R,J)$ *с основным оператором* $T\in\mathfrak{U}_\infty(\mathfrak{H})$ *и такой, что* $\Theta(\zeta)=\Theta_{\mathcal{Y}}(\zeta)$ $(|\zeta|<1)$.

Доказательство. После того, как установлена теорема 5. 1, основная тяжесть доказательства снимается результатами Ю. П. Гинзбурга, указанными выше.

В самом деле, согласно представлению (6. 24) функция $\varphi(\zeta)=I+U_0^{-1}\Theta(\zeta)$ имеет вид $\varphi(\zeta)=2I+V(\zeta)$, где функция $V(\zeta)$ принимает вполне непрерывные значения во всех точках голоморфности функции $\Theta(\zeta)$. Так как $\varphi(0)$ — обратимый оператор, то по теореме И. Ц. Гохберга (см. [28], стр. 39) функция $\Theta(\zeta)$ голоморфна и обратима во всех точках единичного круга, за исключением, быть может, счетного множества точек с предельными точками на единичной окружности или на ε_Θ.

С другой стороны, в силу условия в) функция $\Omega(\zeta)$, определенная равенством (6. 1), удовлетворяет соотношению (6. 3) всюду, за исключением, быть может, особых точек. Учитывая, что условие (6. 3) исключает существование у функции $\Theta(\zeta)$ изолированных особых точек, приходим к выводу, что функция $\Omega(\zeta)$ голоморфна всюду внутри единичного круга.

Таким образом, функция $\Theta(\zeta)\in\mathfrak{A}(J)$ и, значит, существует узел $\mathcal{Y}=(\mathfrak{H},\mathfrak{G};T,R,J)$, такой, что $\Theta(\zeta)=\Theta_{\mathcal{Y}}(\zeta)$. Остается теперь заметить, что в силу условия б) имеем

$$R^*R=J-K^*JK=J-\Theta^*(0)J\Theta(0)\in\mathfrak{S}_\infty$$

и, следовательно, $I-T^*T=RJR^*\in\mathfrak{S}_\infty$. Теорема доказана.

Теорема 6.2. может быть распространена на случай обратимых операторов T, удовлетворяющих условию $I-T^*T\in\mathfrak{S}$, где $\mathfrak{S}$ — какой-либо с.-н. идеал (см. [28]).

Если узел $\mathcal{Y}=(\mathfrak{H},\mathfrak{G};T,R,J)$ таков, что T — сжатие $(J=I)$ и $\mathfrak{G},R,J$ принимают значения, указанные в равенствах (1. 4), то правило восстановления узла по его характеристической функции, полученное в процессе доказательства теоремы 6. 1, переходит в правило, указанное в статье [29].

Для произвольного сжатия (без предположения, что оно обратимо) внутреннюю аналитическую характеристику функции вида (1. 15), а также теорему о том, что вполне неунитарный оператор восстанавливается с точностью до унитарной эквивалентности по своей характеристической функции получили ранее Б. С.-Надь и Ч. Фояш [30, 14]. Последний результат был получен этими авторами с помощью найденной ими теоретико-функциональной модели, представляющей самостоятельный интерес.

Для операторов, не являющихся обязательно сжатиями, но подчиненных другим дополнительным условиям теоремы о единственности оператора (с точностью до унитарной эквивалентности), имеющего данную характеристическую функцию, и конструкциями его восстановления по характеристической функции занимались, начиная с работ М. С. Лившица [1, 2], многие авторы [12, 26, 31, 32].

§ 7. Мультипликативное представление оператор-функций

Теорема 6. 2 вместе с теоремами об умножении Θ-функций [6, 7] и об их мультипликативном представлении [8] позволяют установить предложения о мультипликативном представлении для нового класса оператор-функций из $\mathfrak{A}_\infty(J)$. Имеется в виду множество $\mathfrak{A}_\omega(J)$, состоящее из всех $\Theta(\zeta)\in\mathfrak{A}_\infty(J)$, удовлетворяющих условию:

$$J-\Theta^*(0)J\Theta(0)\in\mathfrak{S}_\omega. \quad ^{12)} \tag{7. 1}$$

Предварительно отметим некоторые свойства функций $\Theta(\zeta)\in\mathfrak{A}_\infty(J)$. Пусть $Y=(\mathfrak{H}, \mathfrak{G}; T, R, J)$ — простой узел, отвечающий на основании теоремы 6. 2 функции $\Theta(\zeta)$.

1°. *Для того, чтобы точка a ($|a|\neq 1$) была регулярной для функции $\Theta(\zeta)$, необходимо и достаточно, чтобы* $1/\bar{a}\,\bar{\in}\,\sigma(T)$.

Достаточность условия очевидна. Необходимость доказывается от противного с использованием известной формулы Ф. Рисса для проектора на корневое подпространство нормального собственного числа оператора. Доказательство опускается ввиду того, что оно сходно с доказательством соответствующего предложения о W-функциях (см. [4], стр. 75).

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[12]) Через $\mathfrak{S}_\omega$ обозначается с.-н. идеал В. И. Мацаева, определяемый как совокупность всех вполне непрерывных операторов $A\in[\mathfrak{G}, \mathfrak{G}]$, таких, что

$$(\|A\|_\omega=)\sum_{j=1}^{\infty}\frac{Sj(A)}{2j-1}<\infty,$$

где $\{S_j(A)\}_{j=1}^\infty$ — последовательность всех собственных чисел оператора $(A^*A)^{\frac12}$, взятых с учетом кратности в порядке убывания (см. [28]).

2°. *Для того, чтобы функция $\Theta(\zeta)$ в своей регулярной точке a принимала обратимые значения, необходимо и достаточно, чтобы* $a \overline{\in} \sigma(T)$.

Это предложение непосредственно следует из предыдущего, если учесть соотношение $\Theta_{\mathscr{Y}}^{-1}(\zeta) = J\Theta_{\mathscr{Y}}^{*}(1/\zeta)J$.

Теорема 7.1. *Пусть $\Theta(\zeta)$ — голоморфная внутри единичного круга оператор-функция, значения которой суть обратимые операторы из* $[\mathfrak{G}, \mathfrak{G}]$.

Если $\Theta(\zeta) \in \mathfrak{A}_\omega(J)$, то она допускает мультипликативное представление

$$\Theta(\zeta) = U \overset{\curvearrowleft}{\int_0^1} \left(I - \frac{e^{i\varphi(t)} + \zeta}{e^{i\varphi(t)} - \zeta} dF(t)\right), \tag{7.2}$$

где U — J-унитарный оператор из $[\mathfrak{G}, \mathfrak{G}]$, $F(t)$ $(0 \leqq t \leqq 1)$ — J-самосопряженная, J-неубывающая, непрерывная оператор-функция со значениями из $[\mathfrak{G}, \mathfrak{G}]$, $\varphi(t)$ $(0 \leqq \varphi(t) \leqq 2\pi)$ — неубывающая, непрерывная слева скалярная функция. Интеграл сходится по норме $\mathfrak{S}_\omega$.

Доказательство. Согласно теореме 6.2 найдется простой узел $\mathscr{Y} = (\mathfrak{H}, \mathfrak{G};\ T, R, J)$ с основным оператором из $\mathfrak{U}_\omega(\mathfrak{H})$ такой, что характеристическая функция $\Theta_{\mathscr{Y}}(\zeta) = \Theta(\zeta)$ $(|\zeta| < 1)$.

В силу предложений 1° и 2° спектр оператора T лежит на единичной окружности. Кроме того, найдется максимальная цепочка $\mathfrak{P} = \{P(t)\}$, разделяющая спектр оператора T.[13])

Известным образом (см., например, [33, 34, 35]) можно построить несущественное расширение $\mathscr{Y}_1 = (\mathfrak{H}_1, \mathfrak{G};\ T_1, R_1, J)$ узла $\mathscr{Y}$ и расширить цепочку $\mathfrak{P}$ до непрерывной цепочки $\mathfrak{P}_1 = \{P(t)\}$ $(0 \leqq t \leqq 1)$, значения которой действуют в $\mathfrak{H}_1$ так, чтобы цепочка $\mathfrak{P}_1$ разделяла спектр оператора T_1. Как указано в § 4, $\Theta_{\mathscr{Y}_1}(\zeta) = \Theta_{\mathscr{Y}}(\zeta)(= \Theta(\zeta))$. Формула (7.1) вытекает теперь из результатов статьи [8].[14])

Заметим, наконец, что *если функция $\Theta(\zeta)$ удовлетворяет всем условиям теоремы* 7.1, *за исключением требования голоморфности и обратимости*

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[13]) Цепочка $\mathfrak{P}$ разделяет спектр оператора T, если $PTP = TP (P \in \mathfrak{P})$ и для любой точки $0 \leqq t \leqq 2\pi$ найдется такой ортопроектор $P_t \in \mathfrak{P}$, что спектр оператора $P_tTP_t|P_t\mathfrak{H}$ лежит на дуге $e^{i\tau}(0 \leqq \tau \leqq t)$, а спектр оператора $(I-P_t)T(I-P_t)|(I-P_t)\mathfrak{H}$ на дуге $e^{i\tau}(t \leqq \tau \leqq 2\pi)$. Для того, чтобы существовала цепочка, разделяющая спектр оператора T, достаточно, чтобы $I - T^*T \in \mathfrak{S}_\omega$ (см. [8]).

[14]) В работе [8] интеграл понимается в специальном смысле. Ввиду того, что в настоящей теореме цепочка $\mathfrak{P}_1$ непрерывна, это ограничение становится излишним. Отсюда же вытекает непрерывность функции $F(t)$.

во всех точках $\zeta(0<|\zeta|<1)$, *то с помощью теоремы умножения* [6, 7] *ее можно представить в виде*

$$(7.3)\quad \Theta(\zeta)=U\overset{1}{\underset{0}{\int^{\curvearrowright}}}\left(I-\frac{e^{i\varphi(t)}+\zeta}{e^{i\varphi(t)}-\zeta}dF(t)\right)\overset{s}{\underset{j=1}{\prod^{\curvearrowright}}}\left[Q_j+\frac{\zeta_j-\zeta}{1-\zeta\bar{\zeta}_j}\frac{|\zeta_j|}{\zeta_j}\mathscr{P}_j\right]\qquad(s\leqq\infty),$$

где $\{\zeta_j\}_1^s$ $(|\zeta_j|\neq 1)$ — последовательность изолированных особых точек $\Theta(\zeta)$, P_j — одномерные J-проекторы и $Q_j=I-P_j$. Всякая точка ζ_j входит в последовательность $\{\zeta_j\}_1^s$ столько раз, какова ее кратность. Бесконечное произведение сходится по норме $\mathfrak{S}_\omega$.

Для случая, когда условие (7. 1) заменяется более узким:

$$(7.4)\qquad J-\Theta^*(0)J\Theta(0)\in\mathfrak{S}_1$$

где $\mathfrak{S}_1$ — с.-н. идеал всех ядерных операторов, формулы (7. 2) и (7. 3) были получены ранее Ю. П. Гинзбургом [36—39] в обобщение результата В. П. Потапова [24]. Кроме того, Ю. П. Гинзбург установил для сжимающих функций ($\|\Theta(\zeta)\|<1$), удовлетворяющих условию (7. 4), трудную теорему о единственности представления (7. 2), явившуюся новой, даже в конечномерном случае [36, 37]. В настоящее время она может быть получена также и для сжимающих функций, удовлетворяющих условию (7. 1)[15]).

Заканчивая настоящую статью, укажем еще на то, что приведенные в ней результаты могут быть распространены на случай операторов T, удовлетворяющих вместо условия обратимости более общему условию, а именно: существует по крайней мере одна точка a $(|a|<1)$, такая, что $a\bar{\in}\sigma(T)$ и $1/\bar{a}\bar{\in}\sigma(T)$.

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Теоремы умножения и деления характеристических функций обратимого оператора

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Настоящая статья является продолжением работы [1]. Приводимые в ней утверждения без доказательства помещены в заметке [2].

Работа состоит из двух разделов. Первый из них носит вспомогательный характер. В нем вводится определение операторного узла, удобное для исследования обратимых операторов, «близких» к унитарным, и изучаются некоторые свойства таких узлов. Получаемые при этом результаты имеют много общих черт с соответствующими результатами теории узлов, построенной для исследования операторов, «близких» к самосопряженным (см. [3]). Более того, при доказательстве часто используются методы, изложенные в монографии [3].

Во втором разделе доказывается теорема умножения характеристических функций произвольных обратимых операторов. В частном случае, когда дефектный оператор $I-T^*T$ конечномерен, эта теорема в матричной форме была доказана А. В. Кужелем [4]. Для сжатия T (не обязательно обратимого) и характеристической функции, определенной равенством

$$\Theta_T(\zeta) = \left(T-\zeta(I-TT^*)^{\frac{1}{2}}(I-\zeta T^*)^{-1}(I-T^*T)^{\frac{1}{2}}\right)\Big|\overline{(I-T^*T)\mathfrak{H}},$$

теорема умножения (в более сложной форме) была получена Б. С.-Надем и Ч. Фояшем [5]. А. В. Кужель распространил их результат на случай произвольных линейных ограниченных операторов, но при этом ему пришлось ввести существенные ограничения на инвариантные подпространства [6]. Ранее теоремы такого типа были получены для расширений изометрических операторов М. С. Лившицем и В. П. Потаповым [7] и Ю. Л. Шмульяном [8].

В статье устанавливается также теорема, обратная в некотором смысле теореме умножения характеристических функций.

§ 1. Определение узлов и действий над ними

1. Определение узла. В работе [1] $\mathscr{Y}$-узлом была названа совокупность гильбертовых пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов $T \in [\mathfrak{H}, \mathfrak{H}]$, $R \in [\mathfrak{G}, \mathfrak{H}]$, $J \in [\mathfrak{G}, \mathfrak{G}]$[1]) таких, что оператор T обратим и

$$I - T^*T = RJR^*, \quad J^* = J, \quad J^2 = I. \tag{1.1}$$

Для определения характеристической функции $\mathscr{Y}$-узла в [1] рассматривался обратимый оператор $K \in [\mathfrak{G}, \mathfrak{G}]$, удовлетворяющий условию

$$J - R^*R = K^*JK. \tag{1.2}$$

Для целей настоящей работы необходимо уточнить определение узла включив в него оператор K. Таким образом, *узлом* будем называть в дальнейшем совокупность

$$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix} \tag{1.3}$$

гильбертовых пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов $T \in [\mathfrak{H}, \mathfrak{H}]$, $R \in [\mathfrak{G}, \mathfrak{H}]$, $J \in [\mathfrak{G}, \mathfrak{G}]$, $K \in [\mathfrak{G}, \mathfrak{G}]$, связанных между собой соотношениями (1. 1) и (1. 2) и таких, что операторы T и K обратимы.

Оператор T называется *основным* оператором узла Δ. Операция построения по данному обратимому оператору $T \in [\mathfrak{H}, \mathfrak{H}]$ пространства $\mathfrak{G}$ и операторов R, J и K ($K^{-1} \in [\mathfrak{G}, \mathfrak{G}]$), связанных между собой соотношениями (1. 1) и (1. 2), называется *включением* оператора T в узел. В работах [1, 9] показано, что *любой обратимый оператор T можно включить в некоторый узел.* При этом, *каковы бы ни были пространства $\mathfrak{H}$, $\mathfrak{G}$ и операторы $T \in [\mathfrak{H}, \mathfrak{H}]$* ($T^{-1} \in [\mathfrak{H}, \mathfrak{H}]$), $R \in [\mathfrak{G}, \mathfrak{H}]$, $J \in [\mathfrak{G}, \mathfrak{G}]$, *удовлетворяющие условиям* (1. 1), *всегда найдется обратимый оператор $K \in [\mathfrak{G}, \mathfrak{G}]$ такой, что выполняется равенство* (1. 2) (см. [1] лемма 1. 1 и [9]).

2. Сопряженные узлы. Рассмотрим узел (1. 3) и положим $S = TRK^{-1}J$ Очевидно, что

$$SJS^* = TR(K^*JK)^{-1}R^*T^*. \tag{1.4}$$

Умножая обе части равенства $R(I - JR^*R) = (I - RJR^*)R$ слева и справа соответственно на операторы $(I - RJR^*)^{-1} = (T^*T)^{-1}$ и $(I - JR^*R)^{-1} = (K^*JK)^{-1}J$, получим

$$(T^*T)^{-1}RJ = R(K^*JK)^{-1}. \tag{1.5}$$

[1]) Символом $[\mathfrak{H}_1, \mathfrak{H}_2]$, где $\mathfrak{H}_1$, $\mathfrak{H}_2$ — гильбертовы пространства, обозначается совокупность всех линейных ограниченных операторов, действующих из $\mathfrak{H}_1$ в $\mathfrak{H}_2$.

На основании (1. 4) и (1. 5)

$$SJS^* = T(T^*T)^{-1}RJR^*T^* = T(T^*T)^{-1}(I-T^*T)T^* = I-TT^*. \tag{1.6}$$

Так как $J-S^*S=J-J(K^*)^{-1}R^*T^*TRK^{-1}J$ и $R^*T^*TR=R^*(I-RJR^*)R=$ $= J-R^*R-(J-R^*R)J(J-R^*R) = K^*JK-K^*JKJK^*JK$, то

$$J-S^*S = KJK^*. \tag{1.7}$$

Ввиду (1. 6) и (1. 7) совокупность

$$\Delta^* = \begin{pmatrix} T^* & S & J & K^* \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix} \quad (S = TRK^{-1}J)$$

представляет собой узел, который будем называть *сопряженным* по отношению к Δ.

Замечая, что в силу (1. 1) и (1. 2)

$$T^*S(K^*)^{-1}J = T^*TR(K^*JK)^{-1}J = (I-RJR^*)R(J-R^*R)^{-1}J =$$
$$= RJ(J-R^*R)(J-R^*R)^{-1}J = R,$$

приходим к равенству

$$(\Delta^*)^* = \Delta. \tag{1.8}$$

3. Произведение узлов. *Произведением* узлов

$$\Delta_1 = \begin{pmatrix} T_1 & R_1 & J & K_1 \\ \mathfrak{H}_1 & & & \mathfrak{G} \end{pmatrix} \quad \text{и} \quad \Delta_2 = \begin{pmatrix} T_2 & R_2 & J & K_2 \\ \mathfrak{H}_2 & & & \mathfrak{G} \end{pmatrix} \tag{1.9}$$

называется совокупность пространств $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \mathfrak{G}$ и операторов

$$T = T_1P_1+T_2P_2-T_1R_1K_1^{-1}JR_2^*P_2, \tag{1.10}$$

$$R = R_1+R_2K_1, \quad J, \quad K = K_2K_1, \tag{1.11}$$

где P_j — ортопроекторы на $\mathfrak{H}_j$ $(j=1, 2)$. Обозначим эту совокупность символом $\Delta_2\Delta_1$ и заметим, что пространство $\mathfrak{H}_1$ инвариантно относительно оператора T.

Лемма 1. 1. *Совокупность $\Delta_2\Delta_1$ является узлом.*

Доказательство. Ввиду (1. 10)

$$\begin{aligned} I-T^*T &= I-(T_1^*P_1+T_2^*P_2-R_2S_1^*P_1)(T_1P_1+T_2P_2-S_1R_2^*P_2) = \\ &= I-T_1^*T_1P_1-T_2^*T_2P_2+R_2S_1^*T_1P_1+T_1^*S_1R_2^*P_2-R_2S_1^*S_1R_2^*P_2, \end{aligned} \tag{1.12}$$

где $S_1 = T_1R_1K_1^{-1}J$. Пользуясь равенством (1. 7), получим

$$\begin{aligned} P_2-T_2^*T_2P_2-R_2S_1^*S_1R_2^*P_2 &= R_2JR_2^*P_2-R_2S_1^*S_1R_2P_2 = \\ &= R_2K_1JK_1^*R_2^*P_2. \end{aligned} \tag{1.13}$$

На основании (1.10)

(1.14) $$P_2R = R_2K_1, \quad P_1R = R_1,$$

и поэтому (1.13) можно переписать в виде

(1.15) $$P_2 - T_2^*T_2P_2 - R_2S_1^*S_1R_2^*P_2 = P_2RJR^*P_2.$$

Рассмотрим оператор

$$T_1^*S_1R_2^*P_2 = T_1^*T_1R_1K_1^{-1}JR_2^*P_2.$$

В силу (1.1) и (1.2)

$$T_1^*S_1R_2^*P_2 = (I - R_1JR_1^*)R_1K_1^{-1}JR_2^*P_2 = R_1J(J - R_1^*R_1)K_1^{-1}JR_2^*P_2 = \\ = R_1JK_1^*R_2^*P_2,$$

откуда, с помощью (1.14), имеем

(1.16) $$T_1^*S_1R_2^*P_2 = P_1RJR^*P_2 \quad \text{и} \quad R_2S_1^*T_1P_1 = P_2RJR^*P_1.$$

Кроме того,

$$P_1 - T_1^*T_1P_1 = R_1JR_1^* = P_1RJR^*P_1.$$

Пользуясь этой формулой, а также формулами (1.12), (1.15) и (1.16) приходим к выводу, что

(1.17) $$I - T^*T = P_1RJR^*P_1 + P_2RJR^*P_2 + P_2RJR^*P_1 + P_1RJR^*P_2 = RJR^*.$$

Применяя снова равенства (1.14), будем иметь

$$J - R^*R = J - R^*P_1R - R^*P_2R = J - R_1^*R_1 - K_1^*R_2^*R_2K_1.$$

Так как $J - R_j^*R_j = K_j^*JK_j$ $(j=1, 2)$ и $K = K_2K_1$, то

(1.18) $$J - R^*R = K_1^*JK_1 - K_1^*R_2^*R_2K_1 = K_1^*(J - R_2^*R_2)K_1 = \\ = K_1^*K_2^*JK_2K_1 = K^*JK.$$

Утверждение леммы вытекает из (1.17) и (1.18).

Лемма 1.2. *Если узлы*

(1.19) $$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix}, \quad \Delta_j = \begin{pmatrix} T_j & R_j & J & K_j \\ \mathfrak{H}_j & & & \mathfrak{G} \end{pmatrix} \qquad (j=1, 2)$$

таковы, что $\Delta = \Delta_2\Delta_1$, *то* $\Delta^* = \Delta_1^*\Delta_2^*$.

Доказательство. Произведение $\Delta_1^*\Delta_2^*$ является совокупностью пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов

(1.20) $$T_1^*P_1 + T_2^*P_2 - T_2^*S_2(K_2^*)^{-1}JS_1^*P_1, \\ S_2 + S_1K_2^*, \quad J, \quad K^* = K_1^*K_2^*,$$

где $S_j = T_jR_jK_j^{-1}J$ и P_j — ортопроектор на $\mathfrak{H}_j$ $(j=1, 2)$. Ввиду (1.10)

$$P_2T^*P_1 = -R_2J(K_1^*)^{-1}R_1^*T_1^*P_1.$$

Отсюда и из (1. 1), (1, 2) вытекает, что

$$T_2^* S_2 (K_2^*)^{-1} J S_1^* P_1 = T_2^* T_2 P_2 (K_2^* J K_2)^{-1} (K_1^*)^{-1} R_1^* T_1^* P_1 =$$
$$= (I - R_2 J R_2^*) R_2 (J - R_2^* R_2)^{-1} (K_1^*)^{-1} R_1^* T_1^* P_1 = -P_2 T^* P_1.$$

Следовательно, оператор (1. 20) равен оператору T^*.

Рассмотрим оператор

$$S_2 + S_1 K_2^* = T_2 R_2 K_2^{-1} J + T_1 R_1 K_1^{-1} J K_2^*. \tag{1.21}$$

Поскольку

$$T_1 R_1 K_1^{-1} J K_2^* = T_1 R_1 K_1^{-1} J (K_1^*)^{-1} K_1^* K_2^* = T_1 R_1 (J - R_1^* R_1)^{-1} K^*$$

и $R_1 (J - R_1^* R_1)^{-1} = (I - R_1 J R_1^*)^{-1} R_1 J$, то

$$T_1 R_1 K_1^{-1} J K_2^* = T_1 (T_1^* T_1)^{-1} R_1 J K^* = (T_1^*)^{-1} R J K^*. \tag{1.22}$$

Ввиду (1. 11) $R_2 = P_2 R K_1^{-1}$ и поэтому

$$T_2 R_2 K_2^{-1} J = T_2 R K_1^{-1} K_2^{-1} J = T_2 R K^{-1} J. \tag{1.23}$$

На основании (1. 21), (1. 22) и (1. 23)

$$S_2 + S_1 K_2^* = T_2 R K^{-1} J + (T_1^*)^{-1} R J K^* = (T_2 R + (T_1^*)^{-1} R J K^* J K) K^{-1} J.$$

Принимая во внимание, что

$$(T_1^*)^{-1} R J K^* J K = (T_1^*)^{-1} R J (J - R^* R) = (T_1^*)^{-1} (I - R J R^*) R =$$
$$= (T_1^*)^{-1} T^* T R = \mathscr{P}_1 T R,$$

получим

$$S_2 + S_1 K_2^* = (T_2 R + P_1 T R) K^{-1} J = T R K^{-1} J = S.$$

Лемма доказана.

Узел (1. 3) называется *простым*[2]), если $\mathfrak{H} = \mathfrak{H}_\Delta$, где

$$\mathfrak{H}_\Delta = \bigvee_{n=-\infty}^{\infty} T^n R \mathfrak{G}.^{3)}$$

Если узел (1. 3) не прост, то пространство $\mathfrak{H} \ominus \mathfrak{H}_\Delta$ инвариантно относительно операторов T и T^*, причем T индуцирует в нем унитарный оператор (см. [1]).

Лемма 1. 3. *Если $\Delta = \Delta_2 \Delta_1$ и узел Δ прост, то узлы Δ_1, Δ_2 — также просты.*

Доказательство аналогично доказательству теоремы 2. 2 работы [3].

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[2]) В работе [2] определение простого узла дано в несколько иной форме. Пользуясь результатами работы [1], легко доказать эквивалентность этих определений.

[3]) Символом $\bigvee_{n \in N} \mathfrak{G}_n$, где $\mathfrak{G}_n$ — подмножества из $\mathfrak{G}$, обозначается замыкание линейной оболочки объединения всех $\mathfrak{G}_n (n \in N)$.

4. Проекция узла. В предыдущем пункте по узлам (1.9) был построен узел $\Delta_2\Delta_1$ так, что пространство $\mathfrak{H}_1$ оказалось инвариантным относительно основного оператора узла $\Delta_2\Delta_1$. Рассмотрим обратную задачу. Пусть задан узел

$$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix}$$

и пусть $\mathfrak{H}_1$ — инвариантное относительно T подпространство такое, что индуцированный в нем оператор T_1 обратим.

Будем искать узлы

$$\Delta_1 = \begin{pmatrix} T_1 & R_1 & J & K_1 \\ \mathfrak{H}_1 & & & \mathfrak{G} \end{pmatrix} \quad \text{и} \quad \Delta_2 = \begin{pmatrix} T_2 & R_2 & J & K_2 \\ \mathfrak{H}_2 & & & \mathfrak{G} \end{pmatrix},$$

удовлетворяющие условию $\Delta = \Delta_2\Delta_1$. Для этого обозначим через P_1 ортопроектор, проектирующий $\mathfrak{H}$ на $\mathfrak{H}_1$, и положим $R_1 = P_1 R$. Заметим, что

$$(1.24) \qquad I - T_1^* T_1 = P_1(I - T^*T)P_1 | \mathfrak{H}_1 = P_1 RJR^* P_1 | \mathfrak{H}_1 = R_1 J R_1^*.$$

На основании леммы 1.1 работы [1] (см. также [9]) найдется обратимый оператор $K_1 \in [\mathfrak{G}, \mathfrak{G}]$ такой, что

$$(1.25) \qquad J - R_1^* R_1 = K_1^* J K_1.$$

Определенная таким образом совокупность Δ_1 в силу равенств (1.24) и (1.25) есть узел. Назовем его *проекцией узла Δ на инвариантное подпространство* $\mathfrak{H}_1$ и обозначим символом $\underset{\mathfrak{H}_1}{\overrightarrow{\mathrm{pr}\,\Delta}}$.

Перейдем к построению узла Δ_2. Пусть $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$, P_2 — ортопроектор на $\mathfrak{H}_2$, $T_2 f = P_2 T f$ $(f \in \mathfrak{H}_2)$ и $R_2 = P_2 R K_1^{-1}$. Тогда

$$R_2 J R_2^* f = P_2 R (K_1^* J K_1)^{-1} R^* P_2 f = P_2 R (J - R_1^* R_1)^{-1} R^* P_2 f =$$
$$= P_2 R (J - R^* P_1 R)^{-1} R^* P_2 f = P_2 (I - RJR^* P_1)^{-1} RJR^* P_2 f \qquad (f \in \mathfrak{H}_2).$$

Легко проверить, что $P_2(I - RJR^*P_1)^{-1} = P_2 - P_2 T^*(T_1^*)^{-1} P_1$. Следовательно,

$$(1.26) \qquad R_2 J R_2^* f = \left(P_2 - P_2 T^* (T_1^*)^{-1} P_1\right)(I - T^*T) P_2 f =$$
$$= P_2 f - P_2 T^* T P_2 f + P_2 T^* P_1 T P_2 f = (I - P_2 T^* P_2 T P_2) f = (I - T_2^* T_2) f.$$

Так как

$$J - R_2^* R_2 = J - (K_1^*)^{-1} R^* P_2 R K_1^{-1} = (K_1^*)^{-1} (K_1^* J K_1 - R^* P_2 R) K_1^{-1}$$

и

$$K_1^* J K_1 - R^* P_2 R = J - R_1^* R_1 - R^* P_2 R = J - R^* P_1 R - R^* P_2 R =$$
$$= J - R^* R = K^* J K,$$

то, полагая $K_2 = KK_1^{-1}$, получим

$$J - R_2^* R_2 = (K_1^*)^{-1} K^* J K K_1^{-1} = K_2^* J K_2. \tag{1.27}$$

Ввиду (1. 26) и (1. 27) совокупность Δ_2 является узлом. *Назовем Δ_2 проекцией узла Δ на ортогональное дополнение $\mathfrak{H}_2$ к инвариантному подпространству* и обозначим символом $\underset{\mathfrak{H}_2}{\overleftarrow{\mathrm{pr}\,\Delta}}$.

Покажем теперь, что $\Delta = \Delta_2 \Delta_1$. Произведение узлов $\Delta_2 \Delta_1$ было определено как совокупность пространств $\mathfrak{H}$, $\mathfrak{G}$ и операторов

$$\begin{gathered} T_1 P_1 + T_2 P_2 - T_1 R_1 K_1^{-1} J R_2^* P_2, \\ R_1 + R_2 K_1, \quad J, \quad K = K_2 K_1. \end{gathered} \tag{1.28}$$

Поскольку $R_2 = P_2 R K_1^{-1}$ и $R_1 = P_1 R$, то

$$\begin{aligned} T_1 R_1 K_1^{-1} J R_2^* P_2 &= T_1 R (K_1^* J K_1)^{-1} R^* P_2 = T_1 R (J - R^* P_1 R)^{-1} R^* P_2 = \\ &= T_1 (I - RJR^* P_1)^{-1} RJR^* P_2. \end{aligned}$$

На основании легко проверяемого равенства $P_1 (I - RJR^* P_1)^{-1} = (T_1^* T_1)^{-1} P_1$ имеем

$$T_1 R K_1^{-1} J K_2^* P_2 = (T_1^*)^{-1} (I - T^* T) P_1 = -P_1 T P_2.$$

Отсюда вытекает, что оператор (1. 28) равен T. Кроме того,

$$R_1 + R_2 K_1 = P_1 R + P_2 R = R,$$

и, следовательно, $\Delta = \Delta_2 \Delta_1$.

Таким образом, доказана.

Теорема 1. 1. *Если T — обратимый оператор из $[\mathfrak{H}, \mathfrak{H}]$, обладающий инвариантным подпространством $\mathfrak{H}_1$, в котором индуцируется обратимый оператор, то существуют узлы*

$$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix},$$

$\Delta_1 = \underset{\mathfrak{H}_1}{\overrightarrow{\mathrm{pr}\,\Delta}}$ *и* $\Delta_2 = \underset{\mathfrak{H}_2}{\overleftarrow{\mathrm{pr}\,\Delta}}$ $(\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1)$, *такие что* $\Delta = \Delta_2 \Delta_1$.

Кроме того, отметим следующее очевидное утверждение: *для любых узлов Δ, Δ_1, Δ_2, связанных между собой равенством $\Delta = \Delta_2 \Delta_1$, справедливы формулы* $\Delta_1 = \underset{\mathfrak{H}_1}{\overrightarrow{\mathrm{pr}\,\Delta}}$, $\Delta_2 = \underset{\mathfrak{H}_2}{\overleftarrow{\mathrm{pr}\,\Delta}}$.

Понятие проекций узла

$$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix}$$

введено так, что данному инвариантному подпространству $\mathfrak{H}_1$ оператора T отвечает бесконечное множество проекций на $\mathfrak{H}_1$ и $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$. Если

$$\Delta_1 = \begin{pmatrix} T_1 & R_1 & J & K_1 \\ \mathfrak{H}_1 & & & \mathfrak{G} \end{pmatrix} = \underset{\mathfrak{H}_1}{\overrightarrow{\mathrm{pr}\,\Delta}} \quad \text{и} \quad \Delta_1' = \begin{pmatrix} T_1' & R_1' & J & K_1' \\ \mathfrak{H}_1 & & & \mathfrak{G} \end{pmatrix} = \underset{\mathfrak{H}_1}{\overrightarrow{\mathrm{pr}\,\Delta}},$$

то $T_1 = T_1'$, $R_1 = R_1'$, и существует J-изометрический оператор $U \in [\mathfrak{G}, \mathfrak{G}]$ такой, что $K_1 = UK_1'$. Если, кроме того,

$$\Delta_2 = \begin{pmatrix} T_2 & R_2 & J & K_2 \\ \mathfrak{H}_2 & & & \mathfrak{G} \end{pmatrix} = \underset{\mathfrak{H}_2}{\overleftarrow{\mathrm{pr}\,\Delta}}, \quad \Delta_2' = \begin{pmatrix} T_2' & R_2' & J & K_2' \\ \mathfrak{H}_2 & & & \mathfrak{G} \end{pmatrix} = \underset{\mathfrak{H}_2}{\overleftarrow{\mathrm{pr}\,\Delta}}$$

и $\Delta = \Delta_2 \Delta_1 = \Delta_2' \Delta_1'$, то $T_2 = T_2'$, $R_2 = R_2' J U^* J$, $K_2 = K_2' J U^* J$.

§ 2. Исследование узла с помощью характеристической функции

1. Теорема умножения. В работах [1, 9] каждому узлу $\mathscr{Y} = (\mathfrak{H}, \mathfrak{G};\ T, R, J)$ была поставлена в соответствие характеристическая функция (Θ-функция)

$$\Theta_{\mathscr{Y}}(\zeta) = J(K^*)^{-1}\left(J - R^*(I - \zeta T^*)^{-1} R\right) \qquad (\zeta \bar{\in} \sigma[(T^*)^{-1}]),^{4})$$

где K-некоторый оператор из $[\mathfrak{G}, \mathfrak{G}]$, удовлетворяющий условию

$$J - R^* R = K^* K. \tag{2.1}$$

Поскольку условием (2. 1) оператор K определен лишь с точностью до левого J-унитарного множителя, то каждому $\mathscr{Y}$-узлу отвечает бесконечное множество Θ-функций. Включение оператора K в узел позволяет установить взаимно однозначное соответствие между узлами и их Θ-функциями.

Итак, *характеристической функцией* (Θ-*функцией*) *узла*

$$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix} \tag{2.2}$$

называется оператор-функция

$$\Theta_{\mathscr{Y}}(\zeta) = J(K^*)^{-1}\left(J - R^*(I - \zeta T^*)^{-1} R\right) \qquad (\zeta \bar{\in} \sigma[(T^*)^{-1}]). \tag{2.3}$$

Теорема 2. 1. *Если узлы*

$$\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix}, \quad \Delta_j = \begin{pmatrix} T_j & R_j & J & K_j \\ \mathfrak{H}_j & & & \mathfrak{G} \end{pmatrix} \qquad (j = 1, 2) \tag{2.4}$$

таковы, что $\Delta = \Delta_2 \Delta_1$, *то*

$$\Theta_{\Delta}(\zeta) = \Theta_{\Delta_2}(\zeta)\Theta_{\Delta_1}(\zeta) \qquad (\zeta \bar{\in} \sigma[(T_1^*)^{-1}] \cup \sigma[(T_2^*)^{-1}]). \tag{2.5}$$

Доказательство.[5]) Обозначим через P_j ортопроектор на $\mathfrak{H}_j$ $(j = 1, 2)$. Как известно,

$$(I - \zeta T^*)^{-1} = P_1 (I - \zeta T_1^*)^{-1} P_1 + P_2 (I - \zeta T_2^*)^{-1} P_2 + \\ + \zeta P_2 (I - \zeta T_2^*)^{-1} P_2 T^* P_1 (I - \zeta T_1^*)^{-1} P_1.$$

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[4]) Символом $\sigma(A)$, где $A \in [\mathfrak{H}, \mathfrak{H}]$, обозначают спектр оператора A.

[5]) При доказательстве теоремы 2.1 обобщаются на бесконечномерный случай некоторые рассуждения А. В. Кужеля [4].

Следовательно,

$$\Theta_\Delta(\zeta) = J(K^*)^{-1}(J-R^*P_1(I-\zeta T_1^*)^{-1}P_1R)- \\ -J(K^*)^{-1}(R^*P_2(I-\zeta T_2^*)^{-1}P_2R+A(\zeta)), \tag{2.6}$$

где $A(\zeta) = \zeta R^*P_2(I-\zeta T_2^*)^{-1}P_2T^*P_1(I-\zeta T_1^*)^{-1}P_1R$.

В силу (1. 11)

$$P_1R=R_1, \quad P_2R=R_2K_1. \tag{2.7}$$

Отсюда вытекает, что

$$J-R^*P_1(I-\zeta T_1^*)^{-1}P_1R = J-R_1^*(I-\zeta T_1^*)^{-1}R_1 = K_1^*J\Theta_{\Delta_1}(\zeta), \\ R^*P_2(I-\zeta T_2^*)^{-1}P_2R = K_1^*R_2^*(I-\zeta T_2^*)^{-1}R_2K_1 = K_1^*(J-K_2^*J\Theta_{\Delta_2}(\zeta))K_1. \tag{2.8}$$

На основании (1. 10)

$$P_2T^*P_1 = -R_2J(K_1^*)^{-1}R_1^*T_1^*. \tag{2.9}$$

Так как $\zeta T_1^*(I-\zeta T_1^*)^{-1} = (I-\zeta T_1^*)^{-1}-I$ и $R_1^*R_1 = J-K_1^*JK_1$, то

$$R_1^*\zeta T_1^*(I-\zeta T_1^*)^{-1}R_1 = R_1^*(I-\zeta T_1^*)^{-1}R_1-R_1^*R_1 = K_1^*JK_1-K_1^*J\Theta_{\Delta_1}(\zeta). \tag{2.10}$$

Пользуясь формулами (2. 7), (2. 9) и (2. 10), получим

$$A(\zeta) = -K_1^*R_2^*(J-\zeta T_2^*)^{-1}R_2J(K_1^*)^{-1}R_1^*\zeta T_1^*(I-\zeta T_1^*)^{-1}R_1 = \\ = -K_1^*(J-K_2^*J\Theta_{\Delta_2}(\zeta))(K_1-\Theta_{\Delta_1}(\zeta)). \tag{2.11}$$

Ввиду соотношений (2. 6), (2. 8) и (2. 9)

$$\Theta_\Delta(\zeta) = J(K^*)^{-1}K_1^*J\Theta_{\Delta_1}(\zeta)-J(K^*)^{-1}[K_1^*(J-K_2^*J\Theta_{\Delta_2}(\zeta))K_1- \\ -K_1^*(J-K_2^*J\Theta_{\Delta_2}(\zeta))(K_1-\Theta_{\Delta_1}(\zeta))].$$

Производя в последнем равенстве элементарные алгебраические преобразования и замечая, что $K=K_2K_1$, придем к утверждению теоремы.

Из теорем 2. 1 и 1. 1 следует

Теорема 2. 2. *Пусть T — обратимый оператор из $[\mathfrak{H}, \mathfrak{H}]$, $\mathfrak{H}_1$ — инвариантное относительно T подпространство такое, что в нем индуцируется обратимый оператор T_1, P_1 — ортопроектор на $\mathfrak{H}_1$, $\mathfrak{H}_2 = \mathfrak{H}\ominus\mathfrak{H}_1$, $P_2 = I-P_1$, $T_2f=P_2Tf$ $(f\in\mathfrak{H}_2)$ и Δ — некоторый узел с основным оператором T.*

Тогда найдутся узлы Δ_1 и Δ_2 с основными операторами T_1 и T_2 соответственно такие, что

$$\Theta_\Delta(\zeta) = \Theta_{\Delta_2}(\zeta)\Theta_{\Delta_1}(\zeta) \qquad (\zeta\,\overline{\in}\,\sigma[(T_1^*)^{-1}]\cup\sigma[(T_2^*)^{-1}]).$$

2. Теорема деления. Пусть $\mathfrak{G}$ — некоторое гильбертово пространство и $J\in[\mathfrak{G}, \mathfrak{G}]$ — сигнатурный оператор ($J^2=I$, $J^*=J$). В работах [1, 9] через $\mathfrak{A}(I)$ был обозначен класс всех оператор-функции $\Theta(\zeta)$ со значениями из $[\mathfrak{G}, \mathfrak{G}]$, удовлетворяющих следующим условиям:

I. $\Theta(\zeta)$ определена и голоморфна в некоторой окрестности точки $\zeta=0$;

II. оператор $\Theta(0)$ является обратимым двусторонним J-сжатием $(\Theta^*(0)J\Theta(0) \leqq J,\ \Theta(0)J\Theta^*(0) \leqq J)$;

III. оператор-функция

$$\Omega(\zeta) = J(I - U_0^{-1}\Theta(\zeta))(I + U_0^{-1}\Theta(\zeta))^{-1},$$

где оператор U_0 взят из J-полярного представления

$$\Theta(0) = U_0 H_0 \qquad (U_0^* J U_0 = U_0 J U_0^* = J,\ (JH_0)^* = JH_0,\ \sigma(H_0) \subset (0, \infty)),$$

допускает аналитическое продолжение на весь круг $|\zeta| < 1$, которое удовлетворяет неравенству $\Omega(\zeta) + \Omega^*(\zeta) \geqq 0$ $(|\zeta| < 1)$.

Кроме того, в работе [1] было доказано, что все Θ-функции принадлежат соответствующим классам $\mathfrak{A}(J)$ и наоборот, для каждой функции $\Theta(\zeta) \in \mathfrak{A}(J)$ найдется простой узел $\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix}$ такой, что в некоторой окрестности точки $\zeta=0$ выполняется равенство $\Theta(\zeta) = \Theta_\Delta(\zeta)$.

Функцию $\Theta_1(\zeta) \in \mathfrak{A}(J)$ назовем *делителем* функции $\Theta_2(\zeta) \in \mathfrak{A}(J)$, если существует функция $\Theta_{21}(\zeta) \in \mathfrak{A}(J)$ такая, что $\Theta_2(\zeta) = \Theta_{21}(\zeta)\Theta_1(\zeta)$. Если, кроме того, произведение простых узлов, отвечающих соответственно функциям $\Theta_1(\zeta)$ и $\Theta_{21}(\zeta)$, также является простым узлом, то функция $\Theta_1(\zeta)$ называется *правильным делителем* функции $\Theta_2(\zeta)$. С помощью понятия унитарной эквивалентности узлов (см. [1, 2]) легко доказывается корректность этого определения.

Включим обратимый оператор $T \in [\mathfrak{H}, \mathfrak{H}]$ в простой узел $\Delta = \begin{pmatrix} T & R & J & K \\ \mathfrak{H} & & & \mathfrak{G} \end{pmatrix}$ и пусть $\mathfrak{H}_1$, $\mathfrak{H}_2$ — такие инвариантные подпространства оператора T, что в них индуцируются обратимые операторы. Положим $\Delta_1 = \underset{\mathfrak{H}_1}{\mathrm{pr}}\,\overrightarrow{\Delta}$, $\Delta_2 = \underset{\mathfrak{H}_2}{\mathrm{pr}}\,\overrightarrow{\Delta}$.

Теорема 2.3. *Для того чтобы $\mathfrak{H}_1 \subset \mathfrak{H}_2$, необходимо и достаточно, чтобы функция $\Theta_{\Delta_1}(\zeta)$ была правильным делителем функции $\Theta_{\Delta_2}(\zeta)$. При этом $\mathfrak{H}_1 = \mathfrak{H}_2$ тогда и только тогда, когда существует J-унитарный оператор $U \in [\mathfrak{G}, \mathfrak{G}]$ такой, что $\Theta_{\Delta_1}(\zeta) = U\Theta_{\Delta_2}(\zeta)$.*[6])

Доказательво аналогично доказательству теоремы 5.6 работы [3].

В заключение отметим, что, как будет показано в статье [10], теория узлов и их характеристических функций допускает дальнейшее обобщение, при котором будет снято требование обратимости основного оператора.

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[6]) Оператор U принадлежит классу $\mathfrak{A}(J)$ и является характеристической функцией зула $\Delta_0 = \begin{pmatrix} T_0 & R_0 & J & K_3 \\ \mathfrak{H}_0 & & & \mathfrak{G} \end{pmatrix}$, где $\mathfrak{H}_0 = \{0\}$, $R_0 g = 0$ $(g \in \mathfrak{G})$ и $K_0 = U$.

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Vecteurs cycliques et commutativité des commutants

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Introduction

Pour un opérateur (linéaire, borné) T dans l'espace de Hilbert $\mathfrak{H}$ on désignera par $(T)'$ le commutant de T, c'est-à-dire l'ensemble des opérateurs qui permutent à T. Envisageons les trois propriétés suivantes possibles de T:

(i) T admet un vecteur cyclique;
(i_*) T^* admet un vecteur cyclique;
(ii) $(T)'$ est commutatif.

Dans le cas d'un espace $\mathfrak{H}$ de dimension finie, ces propriétés sont équivalentes, et il en est de même des opérateurs normaux dans un espace $\mathfrak{H}$ de dimension quelconque. Ces faits sont bien connus. Un résultat récent est que ces propriétés sont équivalentes aussi pour les opérateurs (en général non normaux) de classe C_0;[1]) *cf.* [3].

D'autre part, on sait que si S est une translation unilatérale simple (p. ex. la multiplication par e^{it} dans l'espace H^2 du cercle unité), l'opérateur $T = S^* \oplus S^*$ vérifie (i) sans vérifier (i_*) (*cf.* [1], section 126), et il est manifeste que T ne vérifie pas (ii). Ainsi, en général, (i) n'entraîne pas ni (i_*) ni (ii). (Mais il est un problème ouvert de savoir si (i) & (i_*) entraîne (ii)). Le fait que, en général, (ii) n'entraîne pas (i), vient d'être démontré par J. A. DEDDENS: Intertwining analytic Toeplitz operators (preprint).

Dans la présente Note on étudiera des cas nouveaux où (i) entraîne (i_*) ou (ii). Une partie de ces résultats a été annoncée au Symposium sur la Théorie des Opérateurs, 1—5. juin 1970, Bloomington, Indiana; *cf.* [4].

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[1]) Un opérateur T est de classe C_0 s'il est une contraction complètement non-unitaire et telle que $u(T)=O$ pour une fonction u analytique et bornée dans le disque unité ouvert, $u \not\equiv 0$.

Opérateurs de classe C_1.

Théorème 1. *Pour un opérateur T de classe $C_{1.}$*[2]) *la propriété* (i) *entraîne les propriétés* (i_*) *et* (ii) *dans chacun des cas suivants:*

(a) *il y a du moins un point dans l'intérieur du cercle unité qui n'est pas une valeur propre de T^*;*

(b) *T est complèment non-unitaire (c. n. u.).*[3])

Remarque. En échangeant les rôles de T et T^* on obtient le dual de ce théorème. Nous allons donner la démonstration pour cette forme duale du théorème, équivalente.

Démonstration.

1. Commençons par quelques faits subsistant pour une contraction quelconque T dans $\mathfrak{H}$.

Soit U_+ la dilation isométrique minimum de T, opérant dans un espace $\mathfrak{K}_+(\supset\mathfrak{H})$. On a

(1) $$TP_{\mathfrak{H}} = P_{\mathfrak{H}}U_+ \quad \text{et} \quad \bigvee_{n\geq 0} U_+^n\mathfrak{H} = \mathfrak{K}_+$$

où $P_{\mathfrak{H}}$ désigne la projection orthogonale de $\mathfrak{K}_+$ au sous-espace $\mathfrak{H}$.

Soit $\mathfrak{R}$ le sous-espace de $\mathfrak{K}_+$ dans lequel U_+ est unitaire:

(2) $$\mathfrak{R} = \bigcap_{n\geq 0} U_+^n\mathfrak{K}_+.$$

En désignant par $P_{\mathfrak{R}}$ la projection orthogonale de $\mathfrak{K}_+$ sur le sous-espace $\mathfrak{R}$, on a

(3) $$P_{\mathfrak{R}}h = \lim_{n\to\infty} U_+^n T^{*n}h \quad \text{pour tout} \quad h\in\mathfrak{H};$$

cf. proposition II. 3. 1 dans [2]. Il en dérive que

(4) $$U_+P_{\mathfrak{R}}T^*h = P_{\mathfrak{R}}h \quad \text{pour tout} \quad h\in\mathfrak{H}.$$

Les opérateurs

(5) $$X = P_{\mathfrak{R}}|\mathfrak{H}\ (\mathfrak{H}\to\mathfrak{R}) \quad \text{et} \quad X^* = P_{\mathfrak{H}}|\mathfrak{R}\ (\mathfrak{R}\to\mathfrak{H})$$

sont évidemment adjoints l'un à l'autre. En multipliant dans (4) par l'adjoint de l'opérateur unitaire

(6) $$R = U_+|\mathfrak{R},$$

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[2]) La classe $C_{1.}$ est constituée des contractions T telles que T^nh ne tend fortement vers 0 (lorsque $n\to\infty$) pour aucun $h\in\mathfrak{H}$, $h\neq 0$. La classe $C_{.1}$ est le dual de la classe $C_{1.}$: elle est constituée des contractions T telles que $T^{*n}h$ ne tend vers 0 pour aucun $h\in\mathfrak{H}$, $h\neq 0$. On pose $C_{11}=C_{1.}\cap C_{.1}$.

[3]) Il suffit aussi de supposer *seulement* que la partie unitaire de T ait son spectre absolument continu.

on obtient

$$XT^* = R^*X. \tag{7}$$

En posant

$$\mathfrak{R}_0 = \overline{X\mathfrak{H}} \quad (=\overline{P_{\mathfrak{R}}\mathfrak{H}}) \tag{8}$$

et

$$V = R^* | \mathfrak{R}_0, \tag{9}$$

il s'ensuit de (7) que

$$V\mathfrak{R}_0 \subset \mathfrak{R}_0, \tag{10}$$

et de (1), (6) et (8) il s'ensuit que

$$\mathfrak{R} = P_{\mathfrak{R}}\mathfrak{K}_+ = \bigvee_{n\geqq 0} P_{\mathfrak{R}}U_+^n\mathfrak{H} = \bigvee_{n\geqq 0} U_+^n P_{\mathfrak{R}}\mathfrak{H} = \bigvee_{n\geqq 0} R^n\mathfrak{R}_0. \tag{11}$$

V est donc une isométrie dans $\mathfrak{R}_0$ et R^* est un prolongement unitaire minimum de V.

Notons la conséquence de (1), (4) et (5):

$$X^*X = P_{\mathfrak{H}} \lim_{n\to\infty} U_+^n T^{*n} = \lim_{n\to\infty} T^n T^{*n}; \tag{12}$$

il s'ensuit que

$$\|Xh\|^2 = \lim_{n\to\infty} \|T^{*n}h\| \quad \text{pour tout} \quad h\in\mathfrak{H}. \tag{13}$$

Cela montre que si $T\in C_{\cdot 1}$, on a $Xh=0$ pour $h=0$ seulement et que par conséquent

$$\overline{X^*\mathfrak{R}} = \mathfrak{H}. \tag{14}$$

La relation (7) entraîne pour $n=0, 1, \ldots$

$$a)\quad XT^{*n} = R^{*n}X \quad \text{et} \quad b)\quad T^nX^* = X^*R^n. \tag{15}$$

Montrons, toujours dans l'hypothèse $T\in C_{\cdot 1}$, que si R admet un vecteur cyclique r (dans $\mathfrak{R}$), T admet un vecteur cyclique X^*r (dans $\mathfrak{H}$) et $(T)'$ est commutatif.

La première assertion est une conséquence immédiate de (15b) et (14).

Dans la démonstration de la seconde assertion on observe d'abord que l'existence du vecteur cyclique r pour l'opérateur unitaire R entraîne que $(R)'$ est commutatif. Ensuite on applique le théorème sur la dilatation des commutants (*cf.* [2], version anglaise, théorème II. 2. 3): A tout $A\in(T)'$ on peut attacher un $B\in(U_+)'$ tel que $A = P_{\mathfrak{H}}B_{\mathfrak{H}}|\mathfrak{H}$, $P_{\mathfrak{H}}B(I-P_{\mathfrak{H}}) = 0$ et par conséquent

$$AP_{\mathfrak{H}} = P_{\mathfrak{H}}BP_{\mathfrak{H}} = P_{\mathfrak{H}}B. \tag{16}$$

De plus on a

$$B\mathfrak{R} = B\bigcap_{n\geqq 0} U_+^n\mathfrak{K}_+ \subset \bigcap_{n\geqq 0} BU_+^n\mathfrak{K}_+ = \bigcap_{n\geqq 0} U_+^n B\mathfrak{K}_+ \subset \bigcap_{n\geqq 0} U_+^n\mathfrak{K}_+ = \mathfrak{R}.$$

Donc $C = B|\mathfrak{R}$ est un opérateur dans $\mathfrak{R}$ et il est manifeste que $B \in (U_+)'$ entraîne $C \in (R)'$. De plus, on déduit de (16) et (5) que

$$AX^* = X^* C. \tag{17}$$

Soient $A_1, A_2 \in (T)'$ et soient $C_1, C_2 \in (R)'$ les opérateurs y attachés de cette manière. En appliquant (17) on obtient

$$A_2 A_1 X^* = A_2 X^* C_1 = X^* C_2 C_1 = X^* C_1 C_2 = A_1 X^* C_2 = A_1 A_2 X^*.$$

Vu (14) on en déduit que $A_2 A_1 = A_1 A_2$.

2. Pour achever la démonstration du théorème pour un $T \in C_{\cdot 1}$ tel que T^* admet un vecteur cyclique h_*, il n'y a donc qu'à chercher dans quelles conditions additionnelles l'opérateur correspondant R a-t-il un vecteur cyclique.

Observons d'abord que (15*a*) entraîne que le vecteur $r_0 = Xh_*$ est cyclique pour l'opérateur V dans $\mathfrak{R}_0$, définis par (8) et (9).

Dans le cas où $\mathfrak{R}_0$ coïncide avec l'espace entier $\mathfrak{R}$, on a $V = R^*$. Or, R^* étant unitaire, le vecteur cyclique $r_0 = Xh_*$ pour R^* est cyclique pour R aussi.[4])

Ainsi, dans le cas $\mathfrak{R}_0 = \mathfrak{R}$ tout est démontré.[5]) On sait que ce cas se présente toujours quand les valeurs propres de T ne recouvrent pas tout l'intérieur du cercle unité (*cf.* [2], proposition II. 3. 2).

Passons au cas où $\mathfrak{R}_0 \neq \mathfrak{R}$. Dans ce cas, l'opérateur V n'est pas unitaire.

Supposons que la partie unitaire de la contraction $T(\in C_{\cdot 1})$ a son spectre absolument continu. Il en est alors de même de la dilatation unitaire minimum U de T (conséquence de [2], théorème II. 6. 4) ainsi que de la relation $R^* = U^*|\mathfrak{R}$.

Soit $\{E_t\}$ la famille spectrale attachée à R^* et posons

$$\alpha(t) = \frac{d}{dt}(E_t r_0, r_0).$$

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[4]) Cela subsiste même pour tout opérateur normal N. En effet, si N admet un vecteur cyclique v, il existe pour tout entier $\nu \geqq 0$ et tout $\varepsilon > 0$ un polynôme $p(\lambda) = \sum_0^n c_m \lambda^m$ tel que pour $M = N^{*\nu} - p(N)$ on ait $\|Mv\| < \varepsilon$. Comme M est normal, on a $\|M^* v\| = \|Mv\|$, d'où

$$\|N^\nu v - q(N^*) v\| < \varepsilon \quad \text{où} \quad q(\lambda) = \sum_0^n \bar{c}_m \lambda^m.$$

Cela montre que le sous-espace déterminé par les vecteurs $N^{*\mu} v$ $(\mu = 0, 1, \ldots)$ comprend les vecteurs $N^\nu v$ $(\nu = 0, 1, \ldots)$, donc coïncide avec l'espace entier. Ainsi, v est cyclique pour N^* aussi.

[5]) T admet alors le vecteur cyclique $h = X^* r_0 = X^* X h_*$. Eu égard à (12) on obtient donc que dans le cas en question tout vecteur cyclique h_* pour T^* engendre un vecteur cyclique h pour T moyennant la relation $h = \lim_{n \to \infty} T^n T^{*n} h_*$.

Pour $n \geqq m \geqq 0$ on a

$$(V^n r_0, V^m r_0) = (V^{n-m} r_0, r_0) = (R^{*n-m} r_0, r_0) = \int_0^{2\pi} e^{i(n-m)t} \alpha(t)\, dt.$$

On en déduit que

$$\left\| \sum_0^n c_m V^m r_0 \right\|^2 = \int_0^{2\pi} \left| \sum_0^n c_m e^{imt} \right|^2 \alpha(t)\, dt \tag{18}$$

pour des coefficients c_m quelconques. Puisque r_0 est cyclique pour V et que V n'est pas unitaire, r_0 ne peut appartenir à $V\mathfrak{R}_0$, donc r_0 a une distance positive d à $V\mathfrak{R}_0$. Mais on déduit de (18) que

$$d^2 = \inf_p \int_0^{2\pi} |1 + p(e^{it})|^2 \alpha(t)\, dt$$

où p parcourt la totalité des polynômes s'annulant à l'origine. D'après un théorème de SZEGŐ (*cf.* [2], n° II. 6. 2) on a donc $\log \alpha(t) \in L^1(0,2\pi)$ et par conséquent il existe une fonction extérieure $u \in H^1$ telle que $\sqrt{\alpha(t)} = |u(e^{it})|$ p. p. (*cf.* p. ex. [2], n° III. 1. 1). En vertu de (18), l'application

$$\sum_0^n c_m V^m r_0 \to \sum_0^n c_m e^{imt} u(e^{it})$$

est isométrique; en la prolongeant par continuité on obtient une application unitaire

$$\tau \colon \mathfrak{R}_0 \to H^2 \text{ (l'espace de Hardy—Hilbert).}$$

(Ici on fait usage de ce que r_0 est cyclique pour V, et du théorème de Beurling que les fonctions extérieures (et celles-ci seulement) son cycliques pour l'opérateur S de multiplication par e^{it} dans H^2.) Comme V est le transformé par l'opérateur unitaire τ^{-1} de S, V est aussi une translation unilatérale simple. Par conséquent, le prolongement unitaire minimum R^* de V est une translation bilatérale simple et il est alors de même pour R.

Or, cela entraîne que R admet un vecteur cyclique. En effet, si l'on représente R par l'opérateur de multiplication par e^{it} dans $L^2(0,2\pi)$, toute fonction $w \in L^2(0,2\pi)$ telle que $w(t) \neq 0$ p. p. et $\log |w(t)| \notin L^1$, est cyclique pour R. Par le théorème de Szegő déjà cité (*cf.* p. ex. [2], n° II. 6. 2) on a alors notamment

$$\inf_p \int_0^{2\pi} |1 + p(e^{it})|^2 |w(t)|^2\, dt = 0$$

(où p parcourt les polynômes s'annulant à l'origine), d'où il s'ensuit que le sous-espace de $L^2(0,2\pi)$ sous-tendu par le système $\{e^{int} w(t)\}_{n=0}^{\infty}$ comprend aussi les fonctions $e^{-int} w(t)$ $(n = 1, 2, \ldots)$; or comme $w(t) \neq 0$ p.p., le système $\{e^{int} w(t)\}_{-\infty}^{\infty}$ sous-tend évidemment l'espace entier $L^2(0,2\pi)$.

Ainsi, dans tous les cas considérés, $\boldsymbol{R}$ admet un vecteur cyclique. Cela achève la démonstration du théorème 1.

Contractions faibles

Une contraction T dans l'espace de Hilbert $\mathfrak{H}$ est appelée *faible* si son spectre ne recouvre pas le disque unité et $I-T^*T$ est de trace finie. Une contraction c. n. u. faible T admet une „décomposition C_0—C_{11}", c'est-à-dire qu'il existe des sous-espaces $\mathfrak{H}_0$ et $\mathfrak{H}_1$ de $\mathfrak{H}$, ultrainvariants pour T [6]) et tels que

(19) $$\mathfrak{H}_0 \vee \mathfrak{H}_1 = \mathfrak{H}, \quad \mathfrak{H}_0 \cap \mathfrak{H}_1 = \{0\},$$

et que

$$T_0 = T|\mathfrak{H}_0 \in C_0 \quad \text{et} \quad T_1 = T|\mathfrak{H}_1 \in C_{11}.$$

Notons par Q_i la projection orthogonale de $\mathfrak{H}$ à $\mathfrak{H}_i$ ($i=0, 1$). On a

(20) $$T^n Q_i = Q_i T^n Q_i, \quad Q_i T^{*n} = Q_i T^{*n} Q_i \qquad (i=0, 1;\ n=0, 1, \ldots).$$

Supposons de plus que T^* admet un vecteur cyclique h. On a alors pour $i=0, 1$:

$$\mathfrak{H}_i = Q_i \mathfrak{H} = Q_i \bigvee_{n \geqq 0} T^{*n} h = \bigvee_{n \geqq 0} Q_i T^{*n} h = \bigvee_{n \geqq 0} Q_i T^{*n} Q_i h;$$

puisque $Q_i T^{*n}|\mathfrak{H}_i = T_i^{*n}$, il s'ensuit que le vecteur $Q_i h$ est cyclique pour T_i^*.

Comme T_0^* est de classe C_0, l'existence d'un vecteur cyclique pour T_0^* entraîne que $(T_0^*)'$ soit commutatif; *cf.* [3]. D'autre part, $(T_1^*)'$ est commutatif d'après le théorème 1. Mais alors $(T_0)'$ et $(T_1)'$ sont aussi commutatifs. Or, cela entraîne que $(T)'$ est aussi commutatif.

En effet, si $A \in (T)'$, on a $A\mathfrak{H}_i \subset \mathfrak{H}_i$ ($i=0, 1$) parce que $\mathfrak{H}_i$ est ultrainvariant pour T. En posant $A_i = A|\mathfrak{H}_i$ on obtient:

$$A_i T_i = AT|\mathfrak{H}_i = TA|\mathfrak{H}_i = T_i A_i,$$

donc A_i permute à T_i ($i=0, 1$). En envisageant encore un $A' \in (T)'$ on aura

$$AA'|\mathfrak{H}_i = A_i A_i' = A_i' A_i = A'A|\mathfrak{H}_i \qquad (i=0, 1)$$

et, grâce à la première relation (19), $AA' = A'A$. Donc $(T)'$ est commutatif.

Vu que chacune des propriétés:

a) T est une contraction faible, c. n. u.,

b) $(T)'$ est commutatif,

entraîne la même propriété pour T^*, nous pouvons énoncer notre résultat dans la forme suivante:

Théorème 2. *Pour toute contraction c. n. u. faible T, telle que T ou T^* admet un vecteur cyclique, $(T)'$ est commutatif.*

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[6]) C'est-à-dire invariants pour T ainsi que pour tout opérateur permutant à T.

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Bi-ideals in associative rings

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Throughout this paper, by a ring A we shall mean an arbitrary associative ring. For the terminology we refer to N. JACOBSON [5], N. H. MCCOY [16] and L. RÉDEI [18]. In analogy to the notion of bi-ideal in semigroups (cf. A. H. CLIFFORD and G. B. PRESTON [3] vol. I) we shall study some properties of bi-ideals in rings.

For the arbitrary subsets X and Y of a ring A by the product XY we mean the additive subgroup of the ring A which is generated by the set of all products xy, where $x \in X$, and $y \in Y$. By a bi-ideal B of a ring A we understand a subring B of A satisfying the following condition:

$$BAB \subseteqq B. \tag{1}$$

Obviously every one-sided (left or right) ideal of A is a bi-ideal, and the intersection of a left and a right ideal of A is also a bi-ideal. We note that the bi-ideals in semigroups are special cases of the (m, n)-ideals introduced by S. LAJOS [7]. He remarked that the set of all bi-ideals of a regular ring is a multiplicative semigroup [10]. Some generalizations of biideals of rings were discussed by F. SZÁSZ [22]. The concept of the bi-ideal of semigroups was introduced by R. A. GOOD and D. R. HUGHES [4]. Interesting particular cases of bi-ideals are the quasi-ideals of O. STEINFELD [19]: A submodule Q of an associative ring A is called a quasi-ideal of A if the following condition holds:

$$QA \cap AQ \subseteqq Q. \tag{2}$$

It is known that the product of any two quasi-ideals is a bi-ideal (cf. S. LAJOS [8]). It may be remarked that in case of regular rings the notions of bi-ideal and quasi-ideal coincide (see S. LAJOS [10]). It was shown by the first named author that there exists semigroup S containing a bi-ideal B which is not a quasi-ideal of S (see. S. LAJOS [13]).

Next we formulate some general properties of bi-ideals in rings. Then we characterize two important classes of associative rings in terms of bi-ideals.

Proposition 1. *The intersection of an arbitrary set of bi-ideals B_λ $(\lambda \in \Lambda)$ of a ring A is again a bi-ideal of A.*

Proof. Set $B = \bigcap_{\lambda \in \Lambda} B_\lambda$. Evidently B is a subring of A. From the inclusions $B_\lambda A B_\lambda \subseteqq B_\lambda$ and $B \subseteqq B_\lambda$ $(\forall \lambda \in \Lambda)$ it follows that

$$BAB \subseteqq B_\lambda A B_\lambda \subseteqq B_\lambda \qquad (\forall \lambda \in \Lambda) \tag{3}$$

and consequently we have

$$BAB \subseteqq B. \tag{4}$$

This proves Proposition 1.

Proposition 2. *The intersection of a bi-ideal B of a ring A and of a subring S of A is always a bi-ideal of the ring S.*

Proof. Let us assume that

$$C = B \cap S. \tag{5}$$

Since S is a subring and $C \subseteqq S$ we conclude

$$CSC \subseteqq SSS \subseteqq S. \tag{6}$$

On the other hand

$$CSC \subseteqq BSB \subseteqq BAB \subseteqq B, \tag{7}$$

whence $CSC \subseteqq B \cap S = C$.

Proposition 3. *For an arbitrary subset T of a ring A and for a bi-ideal B of A the products BT and TB both are bi-ideals of A.*

Proof. By $TA \subseteqq A$ and $BAB \subseteqq B$ we have

$$B(TA)B \subseteqq BAB \subseteqq B. \tag{8}$$

Moreover, we have the following monotonity property of the product defined in the introduction above:

$$X \subseteqq Y \Rightarrow XZ \subseteqq YZ \tag{9}$$

for arbitrary subsets X, Y, Z of the ring A. Then (8) and (9) imply the relation

$$(BT)A(BT) \subseteqq BT, \tag{10}$$

which together with $(BT)(BT) = (BTB)T \subseteqq (BAB)T \subseteqq BT$ means that the product BT is a bi-ideal of the ring A. The proof concerning the product TB is similar to that of BT.

In an analogy to the case of semigroups (cf. S. Lajos [8]) we obtain the following result.

Proposition 4. *Let B be an arbitrary bi-ideal of the ring A, and C be a bi-ideal of the ring B such that $C^2 = C$. Then C is a bi-ideal of the ring A.*

Proof. The suppositions $BAB \subseteqq B$ and $CBC \subseteqq C$ imply

$$CAC = C^2AC^2 \subseteqq C(BAB)C \subseteqq CBC \subseteqq C \tag{11}$$

which proves the statement.

Proposition 5. *An arbitrary associative ring A contains no non-trivial bi-ideal if and only if A either is a zero ring of prime order or A is a division ring.*

Proof. Suppose that the ring A contains no non-trivial bi-ideals. Then clearly A contains no non-trivial right ideals, and thus A satisfies the minimum condition on right ideals. Suppose that A is not semi-simple in the sense of JACOBSON. Then A is an Artinian radical ring, which is nilpotent by a well-known result due to CH. HOPKINS (cf. N. JACOBSON [5]), and finally A is a zero ring of prime order in absence of non-trivial right ideals. On the other hand, if A is semi-simple then it is a division ring by the famous WEDDERBURN—ARTIN structure theorem (cf. JACOBSON [5] or RÉDEI [18]), which proves the "only if" part of Proposition 5.

Conversely assume that A either is a zero ring of prime order or a division ring. We shall show that A has no non-trivial bi-ideals. This assertion is trivially true for a zero ring of prime order because every additive subgroup in a zero ring is a two-sided ideal. If A is a division ring and B is a non-zero bi-ideal of A, then the condition

$$BAB \subseteqq B \tag{12}$$

implies $B = A$, because in a division ring A we have $xA = A = Ax$ for every non-zero element $x \in A$, consequently

$$BAB = B(AB) = BA = A \subseteqq B \subseteqq A. \tag{13}$$

Remark 1. An elementary and short proof of the fact that a ring A containing no non-trivial right ideals either is a zero ring of prime order or a division ring, can be found in a paper of F. SZÁSZ [20].

Proposition 6. *Let T be a non-empty subset of the ring A. Then the bi-ideal of A generated by T is of the form:*

$$T_{(1,1)} = IT + T^2 + TAT, \tag{14}$$

where I denotes the ring of rational integers.

Proof. The verification of the statement is almost trivial and we omit it.

Remark 2. By Proposition 1 the intersection of any set of bi-ideals of a ring A is also a bi-ideal of A, and thus the bi-ideal $T_{(1,1)}$ defined above evidently coincides with the intersection of all the bi-ideals of A containing T.

Remark 3. By Proposition 6 we have:

(i) The principal bi-ideal $(x)_{(1,1)}$ generated by the single element x of A can be represented as follows:

$$(x)_{(1,1)} = Ix + Ix^2 + xAx. \tag{15}$$

(ii) In the particular case of an idempotent element e of the ring A we obtain:

$$(e)_{(1,1)} = eAe. \tag{16}$$

(iii) For an additive subgroup T of A one has:

$$T_{(1,1)} = T + T^2 + TAT. \tag{17}$$

(iv) If S is a subring of the ring A then

$$S_{(1,1)} = S + SAS. \tag{18}$$

Proposition 7. *For any associative ring A denote by $\bar{A}$ the set of all additive subgroups of A, and A_1 the set of all bi-ideals of A. Then $\bar{A}$ and A_1 are semigroups under multiplication of subsets (defined in the introduction of this paper), and A_1 is a two-sided ideal of $\bar{A}$.*

Proof. The statement of this proposition is an immediate consequence of Proposition 3 and the definition given in the introduction for the multiplication of subsets.

Remark 4. The multiplicative semigroup of all non-empty subsets of an arbitrary semigroup was formerly investigated by S. Lajos [8]. He proved that the set of all bi-ideals of a semigroup is a two-sided ideal of the multiplicative semigroup of all non-empty subsets of the semigroup.

Remark 5. J. Calais [2] gave an explicite example for a semigroup having two quasi-ideals whose product fails to be a quasi-ideal. In this connection it may be remarked that one of the authors, S. Lajos [10] proved that for the case of regular rings as well as for regular semigroups the product of any two quasi-ideals is again a quasi-ideal.

For the verification of the interesting fact that every left ideal of a right ideal of an arbitrary associative ring can be represented as a right ideal of a suitable left ideal of the ring, we shall prove the following statement in analogy to a semigroup-theoretical result due to S. Lajos [7].

Theorem 1. *For an arbitrary non-empty subset B of an associative ring the following conditions are pairwise equivalent:*

(I) *B is a bi-ideal of A.*
(II) *B is a left ideal of a right ideal of A.*
(III) *B is a right ideal of a left ideal of A.*

Proof. It is enough to prove that (I) is equivalent to (II), because condition (III) is the left-right dual of (II), therefore the proof of the equivalence of (I) and (III) is similar to that of (I)$\Leftrightarrow$(II).

To show that (I) implies (II), suppose that the subset B is a bi-ideal of the ring A. Let $(B)_r$ be the right ideal of A generated by B. It will be verified that B is a left ideal of the ring $(B)_r$. Indeed, the relations $(B)_r = B+BA$ and $BAB \subseteq B$ imply

$$(B)_r B = (B+BA)B \subseteq B^2+BAB \subseteq B. \tag{19}$$

Conversely, to prove that condition (II) implies (I), assume that the subset B of A is a left ideal of a right ideal R of A. Then the inclusions

$$RA \subseteq R, \quad RB \subseteq B \tag{20}$$

imply

$$BAB \subseteq (RA)B \subseteq RB \subseteq B, \tag{21}$$

which together with the obvious fact that B is a subring of A yields the wished assertion.

In what follows we will be concerned with different properties of bi-ideals in special classes of associative rings. Among other things the characterization of some classes of rings will be given by means of bi-ideals.

Theorem 2. *For an associative ring A the following conditions are mutually equivalent:*

(I) *A is regular.*
(II) *$L \cap R = RL$ for every left ideal L and for every right ideal R of A.*
(III) *For every pair of elements a, b of A, $(a)_r \cap (b)_l = (a)_r(b)_l$.*
(IV) *For any element a of A, $(a)_r \cap (a)_l = (a)_r(a)_l$.*
(V) *$(a)_{(1,1)} = (a)_r(a)_2$ for any element a of A.*
(VI) *$(a)_{(1,1)} = aAa$ for any element a of A.*
(VII) *$QAQ = Q$ for any quasi-ideal Q of A.*
(VIII) *$BAB = B$ for any bi-ideal B of A.*

Proof.[1]) (I)$\Leftrightarrow$(II). This was proved by L. Kovács [6]. It is evident that

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[1]) The equivalence of conditions (I)—(VI) in case of semigroups was proved by Lajos [9], [11].

(II)$\Rightarrow$(III)$\Rightarrow$(IV). The implication (IV)$\Rightarrow$(I) was proved by F. SZÁSZ [21]. Thus we have shown the equivalence of the first four conditions.

(I)$\Rightarrow$(V). Assume, that A is a regular ring. Then the solvability of any equation $axa=a$ implies

$$(a)_r=(ax)_r=axA \tag{22}$$

and

$$(a)_l=(xa)_l=Axa \tag{23}$$

where $(ax)^2=ax$ and $(xa)^2=xa$. Hence

$$(a)_r(a)_l = axA\cdot Axa \subseteqq aAa \tag{24}$$

and we conclude

$$(a)_r(a)_l \subseteqq Ia+Ia^2+aAa = (a)_{(1,1)}. \tag{25}$$

Conversely, by condition (IV), it is obvious that

$$(a)_{(1,1)} \subseteqq (a)_r\cap(a)_l = (a)_r(a)_l, \tag{26}$$

Thus (I) implies (V).

To prove that (V)$\Rightarrow$(I), suppose that the ring A satisfies condition (V). Then we have

$$(a)_{(1,1)} = (a)_r(a)_l \tag{27}$$

for any element a in A. (27) implies

$$a\in(Ia+aA)(Ia+Aa) = Ia^2+aAa+aA^2a = Ia^2+aAa. \tag{28}$$

In other words, there exists a rational integer m and an element $b\in A$, such that

$$a = ma^2+aba = a(ma+ba). \tag{29}$$

For the element $e = ma+ba$ we obtain $a=ae$ and $e^2=e$, whence

$$a = ae^2 = a(ma+ba)^2 = a(m^2a^2+maba+mba^2+baba)\in aAa.$$

This implies (I).

It is easy to show that in case of regular rings we have

$$(a)_r(a)_l=aAa, \tag{30}$$

therefore (I)$\Leftrightarrow$(VI).

(I)$\Leftrightarrow$(VII). This has been proved by J. LUH [15].

(I)$\Rightarrow$(VIII). This follows at once from a result of S. LAJOS [10], Theorem 1, and from the above mentioned assertion of J. LUH.

(VIII$\Rightarrow$(I). If A is a ring satisfying condition (VIII), then it satisfies also (VII), which implies (I).

Therefore Theorem 2 is completely proved.

Theorem 3. *The following fifteen conditions for an associative ring are pairwise equivalent:*

(I) *A is strongly regular.*
(II) *A is a two-sided [2]) regular ring.*
(III) *A is a subcommutative [3]) regular ring.*
(IV) *$B^2=B$ for any bi-ideal B of A.*
(V) *$Q^2=Q$ for any quasi-ideal Q of A.*
(VI) *$RL = L\cap R \subseteqq LR$ for any left ideal L and for any right ideal R of A.*
(VII) *$L\cap R = LR$ for every left ideal L and for every right ideal R of A.*
(VIII) *$L_1\cap L_2 = L_1L_2$ and $R_1\cap R_2 = R_1R_2$ for any left ideals L_1, L_2 and for any right ideals R_1, R_2 of A.*
(IX) *$L\cap T = LT$ and $R\cap T = TR$ for every left ideal L, for every right ideal R, and for every two-sided ideal T of A.*
(X) *A is regular and it is a subdirect sum of division rings.*
(IX) *A is a regular ring with no non-zero nilpotent elements.*
(XII) *$L_1\cap L_2 = L_1L_2$ for any two left ideals of A.*
(XIII) *$R_1\cap R_2 = R_1R_2$ for any two right ideals of A.*
(XIV) *$L\cap T = LT$ for any left ideal L and for any two-sided ideal T of A.*
(XV) *$R\cap T = TR$ for any right ideal R and for any two-sided ideal T of A.*

Proof. (I)$\Leftrightarrow$(II). This was proved in [14].

(II) $\Rightarrow$ (III). Assume that A is a two-sided regular ring. Then every onesided (left or right) ideal of A is a two-sided ideal in A, consequently we have

(31) $$AxA\subseteqq xA \quad \text{and} \quad AxA\subseteqq Ax.$$

The solvability of any equation $aya=a$ ($a\in A$) implies $a\in aA$ and $a\in Aa$, for every $a\in A$, therefore by (31)

(32) $$Ax\subseteqq xA \quad \text{and} \quad xA\subseteqq Ax.$$

Thus we conclude that $xA=Ax$ for every element x in A. This exactly is the (two-sided) subcommutativity of the regular ring A.

(III) $\Rightarrow$ (II). Suppose that A is a (two-sided) subcommutative regular ring. Then every principal right ideal $(a)_r$ of A can be generated by an idempotent element e of A, that is

(33) $$(a)_r=(e)_r=eA, \quad e^2=e.$$

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[2]) An associative ring A is said to be a two-sided (or duo) ring if every one-sided (left or right) ideal of A is a two-sided ideal (cf. e.g. THIERRIN [25]).

[3]) For the definition of subcommutative ring we refer to BARBILIAN [1]: a ring A is called (two-sided) subcommutative if $aA=Aa$ for any $a\in A$.

From condition (III) and Theorem 2 we conclude

$$A(a)_r = A(eA) = eA^2 = eA = (a)_r, \tag{34}$$

whence $(a)_r$ is a two-sided ideal. Consequently an arbitrary right ideal R of A is also a two-sided ideal of the ring A. Similarly it can be proved that every left ideal L of A is also a two-sided ideal in A. Thus we have proved that (II)$\Leftrightarrow$(III).

(I)$\Leftrightarrow$(V). This follows from Theorem 2 of L. Kovács [6] and from authors' Theorem in [14].

Next we show that (IV)$\Leftrightarrow$(V).

The implication (IV)$\Rightarrow$(V) is evident. The converse of this statement is a consequence of the above mentioned result of L. Kovács, and Theorem 1 of S. Lajos [10].

Finally the equivalence of the conditions (VI)—(XV) with each other and with condition (I) was proved [14].

Thus Theorem 3 is proved.

It is known that every regular ring is semisimple in the sense of N. Jacobson. The following assertion characterizes the semisimple rings A in the class of rings with property:

($*$) *The lattice of all right ideals of A is a chain*[4]).

Proposition 8. *For a ring A with property* ($*$) *the following conditions are equivalent:*

(I) *A is semisimple.*

(II) *A is regular.*

(III) *A is strongly regular.*

(IV) *A is direct sum of division rings.*

(V) *A is a division ring.*

Proof. In what follows we assume that the ring A satisfies the condition ($*$). It is easy to see, that Proposition 8 will be proved if we demonstrate the equivalence of (I) and (V), because every class (N) of rings in Proposition 8 contains the class of rings with property $(N+I)$, where $N=$ I, II, III, IV.

Suppose that A is a ring with radical $J=0$. Then the intersection of the modular maximal right ideals R_λ $(\lambda \in \Lambda)$ of A is (0) by N. Jacobson [5], Chapter I. In virtue of property ($*$) and of the maximality of the right ideal R_λ we conclude $R_\lambda = 0$, whence A contains no non-trivial right ideals. Therefore A is a division ring.

Proposition 8 is completely proved.

Remark 6. A subclass of the class of rings with property ($*$) was earlier discussed by E. C. Posner [17]. Moreover, L. A. Skornjakov [24] has obtained some results concerning rings with the left-right dual of property ($*$).

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[4]) Cf. Szász [23].

Remark 7. Let A be the ring of all matrices of type 2×2 over the field with two elements. Then A is a ring with sixteen elements having the property that $BAB=B$ holds for every bi-ideal B of A. Moreover, let B_0 be the bi-ideal generated by the element

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then we obviously have $B_0^2=0\neq B_0$. Evidently A is regular, but not strongly regular and A does not satisfy condition $(*)$.

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Local and residual properties in bicategories

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§ 1

In the theory of universal algebras local and residual properties are well known, and they are to some extent dual properties. It is easy to give a categorical definition of these notions, but category theoretically they are not exactly dual. In universal algebra it is proved that any residual property which is preserved under homomorphic images is local but the categorically dual statement is not true even in such a nice category as that of abelian groups (cf. [1], Exercise 3).

The purpose of this paper is twofold. On the one hand, we give a categorical generalization of this connection between local and residual properties. In this way it becomes clear why the dual statement is not true in universal algebra (the reason is GROTHENDIECK's axiom AB 5). On the other hand, as a possible interpretation of the dual statement, we present concrete categories in which it is true. This dual statement, however, yields well known facts of the general topology; we estimate it essential that such a categorical aspect is able to join quite different branches of mathematics.

In our investigations we shall consider a bicategory satisfying some rather natural additional requirements. In § 2 we shall give a categorical definition of local and residual properties with some cardinality-restrictions. Such a subtle definition is suitable with respect to the topological applications. We present also a lemma which establishes an equivalent formulation of a special case of GROTHENDIECK's axiom AB 5. This lemma will be used in the proof of the Theorem of § 3. § 3 is devoted to proving the categorical generalization of the connection between local and residual properties. In § 4 we give concrete categories in which the dual theorem is true (the category of 0-dimensional compact spaces and that of complete metric spaces with closed continuous mappings). By specialization we obtain e.g. that Lindelöf property is preserved by forming inverse limit of countable inverse systems of complete metric spaces.

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§ 2

In general our terminology is based on MITCHELL's book [8]. With respect to the interpretations of § 4 it is appropriate to use the notion of bicategory due to ISBELL [5] (cf. also SEMADENI [9] and KENNISON [6]).

Let $\mathscr{C}$ be a category. Let $\mathscr{I}$ and $\mathscr{S}$ be classes of morphisms on $\mathscr{C}$. Then $(\mathscr{I}, \mathscr{S})$ is a *bicategory structure* on $\mathscr{C}$ provided that

(B_1) $\mathscr{I}$ and $\mathscr{S}$ are subcategories of $\mathscr{C}$,

(B_2) $\mathscr{I} \cap \mathscr{S}$ consists exactly of all equivalences;

(B_3) The morphisms of $\mathscr{I}$ are monomorphisms and the morphisms of $\mathscr{S}$ are epimorphisms;

(B_4) Every morphism φ can be factored as $\varphi = \varphi_2 \varphi_1$ with $\varphi_1 \in \mathscr{S}$, $\varphi_2 \in \mathscr{I}$, moreover this factorization is unique to within an equivalence in the sense that if $\varphi = \nu\mu$ and $\mu \in \mathscr{S}$, $\nu \in \mathscr{I}$ then there exists an equivalence γ for which $\nu\gamma = \varphi_2$ and $\gamma\varphi_1 = \mu$.

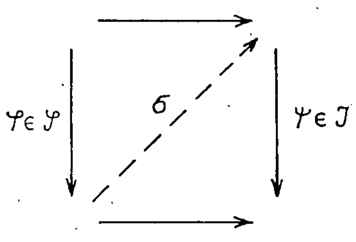
The morphisms of $\mathscr{I}$ and $\mathscr{S}$ are called *injections* and *surjections*, respectively. A category equipped with a bicategorical structure is called briefly a *bicategory*.

Proposition 1 ([6] Prop. 1. 1). *Let $\mathscr{C}$ be a bicategory. Then*

(1) *$\varphi\psi \in \mathscr{I}$ implies $\psi \in \mathscr{I}$;*

(2) *$\varphi\psi \in \mathscr{S}$ implies $\varphi \in \mathscr{S}$;*

(3) *Every commutative diagram of the form indicated by figure (1) can be filled in at σ with commutativity preserved.*



(1)

If $\alpha: A_1 \to A$ is an injection, then A_1 is a *subobject of* A, if $\beta: B \to B_1$ is a surjection, then B_1 is called a *factorobject* of B.

An object S of a category $\mathscr{C}$ is called a *cosingleton*, if the following two conditions are satisfied (cf. SEMADENI [9]):

(i) For every object A there exists exactly one morphism $\alpha: S \to A$;
(ii) For every object B there exists at least one morphism $\beta: B \to S$.

Throughout § 2 and § 3 we shall assume that the considered category $\mathscr{C}$ is a bicategory, further it satisfies the following axioms:

(A_1) $\mathscr{C}$ has a cosingleton;

(A_2) For every family $\{A_i\}_{i \in I}$, $|I| \leqq \aleph$, of factorobjects of any object A in $\mathscr{C}$ the counion $\bigcup^*_{i \in I} A_i$ exists;

(A_3) $\mathscr{C}$ admits products $\prod_{i \in I} A_i$, if $|I| \leqq \aleph$;

(A_4) $\mathscr{C}$ admits direct limits $\varinjlim \{A_i\}_{i \in I}$ if $|I| \leqq \aleph$.

We shall make use of some statements being easy consequences of (A_1) and (A_3).

Suppose that S is a cosingleton, and denote by σ_i the only morphism $S \to A_i$. Let us suppose that *the class of all morphisms* $A_i \to S$ *is a set for every* $A_i \in \mathscr{C}$. Now, by the axiom of choice we can select exactly one morphism $\omega_i: A_i \to S$ for each $A_i \in \mathscr{C}$. Let us define $\omega_{ij} = \omega_i \sigma_j: A_i \to A_j$. By [9], 3,5 for any objects A_i, A_j, A_k we have $\omega_{ij}\omega_{jk} = \omega_{ik}$.

Proposition 2 ([9], 3. 6). *The projection* $\pi_i: \prod_{i\in I} A_i \to A_i$, $|I| \leqq \aleph$, *is a surjection for each* $i \in I$, *and there are injections* $\sigma_i: A_i \to \prod_{i\in I} A_i$ *such that* $\pi_i \sigma_i = 1_{A_i}$, $\pi_j \sigma_i = \omega_{ij}$ *for* $i \neq j$.

Proposition 3. *Consider* $A = \prod_{i\in I} A_i$ *and* $B = \prod_{i\in I} B_i$, $|I| \leqq \aleph$, *with the projections* π_i *and* ϱ_i, $i \in I$, *respectively. If* $\alpha_i: A_i \to B_i$, $i \in I$ *is a family of injections, then there exists a unique injection* $\alpha: A \to B$ *such that* $\varrho_i \alpha = \alpha_i \pi_i$ *holds for each* $i \in I$.

The proof will be analogous to that of [7], § 14. 3 in the case when the cosingleton is a zero object.

By the definition of the product there exists a unique morphism (the so-called canonical morphism) α such that diagram *(2)* is commutative for all $i \in I$. We have to show that α is an injection. Consider a factorization $\alpha = \nu\mu$ with $\mu \in \mathscr{S}$, $\nu \in \mathscr{I}$. By Proposition 1 (3) for each $i \in I$ there exists such a morphism σ_i that diagram *(3)* is commutative. Since $A = \prod_{i\in I} A_i$, the canonical morphism $\gamma: C \to A$

$$\begin{tikzcd} A \arrow[r, "\pi_i"] \arrow[d, "\alpha"'] & A_i \arrow[d, "\alpha_i"] \\ B \arrow[r, "\varrho_i"'] & B_i \end{tikzcd}$$

(2)

$$\begin{tikzcd} A \arrow[r, "\pi_i"] \arrow[d, "\mu"'] & A_i \arrow[d, "\alpha_i"] \\ C \arrow[r, "\varrho_i \nu"'] \arrow[ur, dashed, "\sigma_i"] & B_i \end{tikzcd}$$

(3)

exists. Obviously $\gamma\mu$ has to be 1_A. Hence Proposition 1 (1) implies $\mu \in \mathscr{I}$ and thus α is indeed an injection.

In § 2 and § 3 we shall assume that the bicategory $\mathscr{C}$ satisfies condition

(C) *The direct limit* $\varinjlim \{A_i\}_{i\in I}$ *of every direct family of subobjects* $\{A_i\}_{i\in I}$, $|I| \leqq \aleph$ *of an object is the union* $\bigcup_{i\in I} A_i$.

Condition (C) without any restriction to the cardinality of I, is fulfilled by every category of any primitive class (i.e. variety) of universal algebras (as it turns out e.g. from [3], § 21) and for a complete abelian category (C) is equivalent to GROTHENDIECK's axiom AB 5 (cf. [4]. Proposition 1,8 or [8]. III Proposition 1,2). GROTHENDIECK [4] has also pointed out that a category satisfying axiom AB 5 as well as its dual one, has to consist of zero objects.

We need also an other form of condition (C). Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$, $|I|\leqq\aleph$ be direct systems of subobjects of the objects A and B with $\bigcup\limits_{i\in I} A_i=A$ and $\bigcup\limits_{i\in I} B_i=B$ and with the commutative diagrams *(4* and *5)* for all $i\leqq j\leqq k\in I$.

$$\begin{tikzcd}
A_i \arrow[r, "\alpha_j^i"] \arrow[dr, "\alpha_i"'] & A_j \arrow[r, "\alpha_k^j"] \arrow[d, "\alpha_j"] & A_k \arrow[dl, "\alpha_k"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_i \arrow[r, "\beta_j^i"] \arrow[dr, "\beta_i"'] & B_j \arrow[r, "\beta_k^j"] \arrow[d, "\beta_j"] & B_k \arrow[dl, "\beta_k"] \\
 & B &
\end{tikzcd}$$

(4) *(5)*

Lemma. *Assuming that* $\bigcup\limits_{i\in I} A_i$, $|I|\leqq\aleph$, *exists in* $\mathscr{C}$ *for every direct system of subobjects* $\{A_i\}$ *of any object* A, *condition (C) is equivalent to condition*

(D) *If* φ_i: $B_i\to A_i$, $i\in I$, $|I|\leqq\aleph$ *is a system of surjections such that diagram (6) is commutative for all* $i\leqq j\in I$, *then there exists a unique surjection* φ: $B\to A$ $(B=\bigcup\limits_{i\in I} B_i,\ A=\bigcup\limits_{i\in I} A_i)$ *such that diagram (7) is also commutative for all* $i\in I$.

$$\begin{tikzcd}
B_i \arrow[r, "\beta_j^i"] \arrow[d, "\varphi_i"'] & B_j \arrow[d, "\varphi_j"] \\
A_i \arrow[r, "\alpha_j^i"'] & A_j
\end{tikzcd}
\qquad
\begin{tikzcd}
B_i \arrow[r, "\beta_i"] \arrow[d, "\varphi_i"'] & B \arrow[d, "\varphi"] \\
A_i \arrow[r, "\alpha_i"'] & A
\end{tikzcd}$$

(6) *(7)*

Proof. (C)$\Rightarrow$(D). According to (C) we have diagram *(8)*. Since $\alpha_i\varphi_i$ maps B_i into A such that

$$\alpha_j\varphi_j\beta_j^i = \alpha_j\alpha_j^i\varphi_i = \alpha_i\varphi_i \qquad i \leqq j \in I,$$

therefore by the definition of direct limit there exists a unique morphism, the canonical one, $\varphi: B \to A$ such that $\alpha_i\varphi_i = \varphi\beta_i$ holds for all $i \in I$. Moreover, by (B_4) φ can be

$$\begin{tikzcd}
B_i \arrow[r, "\beta_i"] \arrow[d, "\varphi_i"'] & B = \bigcup\limits_{i\in I} B_i = \varinjlim \{B_i\}_{i\in I} \\
A_i \arrow[r, "\alpha_i"'] & A = \bigcup\limits_{i\in I} A_i = \varinjlim \{A_i\}_{i\in I}
\end{tikzcd}$$

(8)

$$\begin{tikzcd}
B_i \arrow[r, "\mu\beta_i"] \arrow[d, "\varphi\in\mathcal{S}"'] & C \arrow[d, "\nu\in\mathcal{J}"] \\
A_i \arrow[r, "\alpha_i"'] \arrow[ur, dashed, "\sigma_i"] & A
\end{tikzcd}$$

(9)

factored as $\varphi = \nu\mu$ with $\mu \in \mathcal{S}$, $\nu \in \mathcal{J}$ and so according to Proposition 1 (3) for each $i \in I$ there exists a morphism σ_i with commutativity preserved in diagram *(9)*. Hence by the definition of the direct limit there exists the canonical morphism γ such that diagram *(10)* is commutative for all $i \in I$.

$$\begin{tikzcd}
B_i \arrow[r, "\beta_i"] \arrow[d, "\varphi_i"'] & B \arrow[d, "\varphi"] \arrow[dr, "\mu"] & \\
A_i \arrow[r, "\alpha_i"] \arrow[drr, "\sigma_i"'] & A \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\nu"]
\end{tikzcd}$$

(10)

Since $A = \varinjlim \{A_i\}_{i\in I}$, therefore $\nu\gamma = 1_A$ follows. Thus by Proposition 1 (2) we have $\nu \in \mathcal{S}$. This implies $\varphi = \nu\mu \in \mathcal{S}$.

(D)$\Rightarrow$(C). Put $A_i = B_i$, $i \in I$. Now by condition (D) there exists a morphism $\psi\colon \varinjlim \{A_i\} = A \to \bigcup_{i\in I} A_i = A$ and $\varphi\colon \varinjlim \{A_i\} = B \to \bigcup A_i = B$ such that diagrams *(11)* and *(12)* are commutative for all $i \in I$. By the uniqueness of φ and ψ it follows $\varphi\psi = 1_A$. Hence $\psi\colon \varinjlim \{A_i\}_{i\in I} \to \bigcup_{i\in I} A_i$ is an equivalence, and so condition (C) is satisfied.

$$\begin{tikzcd}
 & A = \varinjlim \{A_i\}_{i\in I} \arrow[d, "\psi"] \\
A_i \arrow[ur, "\alpha_i"] \arrow[r] \arrow[dr, "\alpha_i"'] & B = \bigcup_{i\in I} A_i \arrow[d, "\varphi"] \\
 & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & B \arrow[d, "\varphi"] \\
A_i \arrow[ur, "\beta_i"] \arrow[r] \arrow[dr, "\beta_i"'] & A = \bigcup_{i\in I} A_i \arrow[d, "\psi"] \\
 & B
\end{tikzcd}$$

(11) *(12)*

Consider an abstract property **P** of objects of $\mathscr{C}$, i.e. if A and B are equivalent objects, then either both A and B or none of them has property **P**. Since property **P** divides the objects of $\mathscr{C}$ into two classes, so the fact A has property **P** will be denoted by $A \in \mathbf{P}$.

Let $\aleph$ be a cardinality. By an $\aleph$-*local system* of subobjects of an object A one understands a direct system $\{A_i\}_{i\in I}$ such that $\bigcup_{i\in I} A_i = A$ and $|I| \leqq \aleph$. The object A is said to be $\aleph$-*locally* **P**, if there is an $\aleph$-local system of subobjects of A all A_i belonging to **P**. If every object which is $\aleph$-locally **P** actually belongs to **P** itself, then **P** is said to be an $\aleph$-*local property*. In view of condition (C), an $\aleph$-local property **P** means such an abstract property which is closed under forming direct limits of direct systems having cardinality $\leqq \aleph$.

We define an $\aleph$-*residual system* of an object A to be a system $\{A_i\}_{i\in I}$ consisting of factorobjects of A such that $\bigcup^*_{i\in I} A_i = A$ and $|I| \leqq \aleph$. By an $\aleph$-*residual* **P** object we mean an object which has an $\aleph$-residual system consisting of factorobjects belonging to **P**. The property **P** is said to be an $\aleph$-*residual property*, if every object which is $\aleph$-residually **P** actually belongs to **P** itself.

Let us mention that $\aleph$-local and $\aleph$-residual properties are not dual notions.

However, any system $\{A_i\}_{i\in I}$, $|I|\leqq\aleph$, of subobjects of an object A with $\bigcup\limits_{i\in I} A_i=A$ generates a direct system consisting of finite unions $\bigcup\limits_{\text{finite}} A_k$, but $A_i\in\mathbf{P}$, $i\in I$, do not imply $\bigcup\limits_{\text{finite}} A_k\in\mathbf{P}$.

§ 3

In this section we are going to prove the following

Theorem. *Let the bicategory $\mathscr{C}$ satisfy axioms* (A_1)—(A_4) *and condition (C). If* $\mathbf{P}$ *is an* $\aleph$*-residual property preserved by surjections then* $\mathbf{P}$ *is an* $\aleph$*-local property.*

Let us remark that this theorem is valid for any category of Ω-algebras. The corresponding statement without any restrictions to the cardinality, is just Proposition 7, 4 of Cohn [1] (there the existence of cosingleton is not supposed).

Proof. The outline of the proof is the following. We shall consider an object A which is $\aleph$-locally $\mathbf{P}$ with an $\aleph$-local system $\{A_i\}_{i\in I}$, $A_i\in\mathbf{P}$, $|I|\leqq\aleph$. From $\{A_i\}_{i\in I}$ we construct an object B which is $\aleph$-residually $\mathbf{P}$, and so by the assumption B will have property $\mathbf{P}$. Further we shall show that there exists a surjection $B\to A$. Hence also the object A will have property $\mathbf{P}$.

Consider an object A having an $\aleph$-local system $\{A_i\}_{i\in I}$ with injections $\alpha_i\colon A_i\to A$, $\alpha_j^i\colon A_i\to A_j$ such that $\alpha_j\alpha_j^i=\alpha_i$, $i\leqq j\in I$, $|I|\leqq\aleph$, and $A_i\in\mathbf{P}$ for all $i\in\mathbf{P}$. Let $\xi_i\colon A\to D_i$ be an equivalence for all $i\in I$, and form the product $C=\prod\limits_{i\in I} D_i$ with the projections $\pi_i\colon C\to D_i$. By Proposition 2, every π_i is a surjection. Further, for all $i\in I$, define C_i and C_i^* by $C_i=\prod\limits_{i\leqq j\in I} D_j$ and $C_i^*=\prod\limits_{i>j\in I} D_j$, respectively. (For the empty set $\emptyset$ $\prod\limits_{j\in\emptyset} D_j$ means cosingleton.) According to Proposition 3 both C_i and C_i^* are subobjects of C. The object A_i can be embedded "diagonally" in C_i for any $i\in I$ as follows. The morphism $\delta_j^i=\xi_j\alpha_j^i\colon A_i\to D_j$ embeds A_i into D_j for every $i\leqq j\in I$. The canonical morphism $\delta_i\colon A_i\to C_i$ satisfies $\pi_j'\delta_i=\delta_j^i$ where π_j' is the projection $C_i\to D_j$. Hence Proposition 1 (1) implies that δ_i is an injection for each $i\in I$. According to Proposition 3 $B_i=C_i^*\times A_i$ is a subobject of $C=C_i^*\times C_i\times\prod\limits_{i\parallel k\in I} D_k$ by an injection γ_i such that diagram *(13)* is commutative. Here φ_i and ψ_i denote the projections of B_i and C into A_i and C_i, respectively. Moreover by Proposition 2 they are surjections.

$$\begin{tikzcd} B_i \arrow[r, "\varphi_i"] \arrow[d, "\gamma_i"'] & A_i \arrow[d, "\delta_i"] \\ C \arrow[r, "\psi_i"'] & C_i \end{tikzcd}$$

(13)

For any fixed $i<j\in I$, consider the injections $\xi_k\alpha_k^i\colon A_i\to D_k$ $(i\leqq k<j)$ and $\alpha_j^i\colon A_i\to A_j$. Now by Proposition 1 (1) the canonical morphism of A_i into $B_j=$

$= C_i^* \times \prod_{i \leqq k < j} D_k \times A_j$ is an injection for $i < j \in I$. So applying Proposition 3 to B_i and B_j we obtain that there exists a unique injection $\beta_j^i : B_i \to B_j$ such that *(14)* and *(15)* are commutative diagrams for $i \leqq k < j \in I$. Hence letting $\beta_i^i = 1_{B_i}$ for any $i \leqq k \leqq j$ we get $\varphi_j \beta_j^i = \alpha_j^i \varphi_i = \alpha_j^k \alpha_k^i \varphi_i = \alpha_j^k \varphi_k \beta_k^i = \varphi_j \beta_j^k \beta_k^i$, and by the uniqueness of β_j^i we have $\beta_j^i = \beta_j^k \beta_k^i$. Thus $\{B_i\}_{i \in I}$ forms a direct system. With respect to condition (C) we have $B = \underrightarrow{\lim} \{B_i\}_{i \in I} = \bigcup_{i \in I} B_i$, and so B is a subobject of C by injection β. Now $\pi_i \beta$ maps B into D_i. Since D_i is a subobject of B_{i+1} and so of B by an injection δ_i, therefore by Proposition 2 for the injection $\delta_i : D_i \to B$ we have

$$\beta \delta_i = \sigma_i \quad \text{and} \quad \pi_i \beta \delta_i = \sigma \delta_i = 1_{D_i}.$$

Thus by Proposition 1 (2) the morphism $\pi_i \beta : B \to D_i$ is a surjection for all $i \in I$.

$$\begin{tikzcd} B_i \arrow[r, "\varphi_i"] \arrow[d, "\beta_j^i"'] & A_i \arrow[d, "\xi_k \alpha_k^i"] \\ B_j \arrow[r, "\pi_k \gamma_j"'] & D_k \end{tikzcd} \qquad \begin{tikzcd} B_i \arrow[r, "\varphi_i"] \arrow[d, "\beta_j^i"'] & A_i \arrow[d, "\alpha_j^i"] \\ B_j \arrow[r, "\varphi_j"'] & A_j \end{tikzcd}$$

(14) *(15)*

Now we are able to prove $B \in \mathbf{P}$. To this aim it is sufficient to show $\bigcup_{i \in I}^* D_i = B$ because $\mathbf{P}$ is an $\aleph$-residual property and $\pi_i \beta : B \to D_i \approx A_i \in \mathbf{P}$ is surjection for all $i \in I$. Put $B_0 = \bigcup_{i \in I}^* D_i$. Now there exist surjections $\beta_0 : B \to B_0$ and $\varrho_i : B_0 \to D_i$ such that $\varrho_i \beta_0 = \pi_i \beta$ is valid for all $i \in I$. On the other hand B_0 can be embedded in $C = \prod_{i \in I} D_i$ by the canonical morphism ϱ_0 such that $\pi_i \varrho_0 = \varrho_i$ holds for every $i \in I$. Hence we have $\pi_i \varrho_0 \beta_0 = \pi_i \beta$ and the uniqueness of β yields $\varrho_0 \beta_0 = \beta$. Since β is an injection, so by Proposition 1 (1) also β_0 is an injection. Hence $\beta_0 \in \mathcal{S} \cap \mathcal{I}$, and so B and B_0 are equivalent objects. Thus $B \in \mathbf{P}$ is proved.

By (*I*) we have $\alpha_j^i \varphi_i = \varphi_j \beta_j^i$ for all $i \leqq j \in I$. Thus with respect to the Lemma there exists a surjection $\varphi : B \to A$ such that $\varphi \beta_i = \alpha_i \varphi_i$ is valid for all $i \in I$. Since property $\mathbf{P}$ is preserved by surjections, so $B \in \mathbf{P}$ implies $A \in \mathbf{P}$. Hence $\mathbf{P}$ is $\aleph$-local, and the theorem is proved.

§ 4

1. Let $\mathscr{C}_B$ be the category of Boolean algebras. In $\mathscr{C}_B$ all the conditions (A_1)—(A_4) as well as (C) are satisfied without any restriction on the cardinality. By the well-known duality between the category $\mathscr{C}_B^*$ of 0-dimensional compact spaces (the so-called Boolean spaces) and that of Boolean algebras, the dual statement of the Theorem holds in $\mathscr{C}_B^*$. (According to the duality we hint to [9].)

2. As an other possibility to interpret the dual statement of the Theorem, let us consider the category $\mathscr{C}_M$ consisting of complete metric spaces (with bounded metric) and closed continuous mappings. For the notions well known in general topology' we refer to ENGELKING's book [2]. $\mathscr{C}_M$ becomes a bicategory by choosing $\mathscr{I}$ and $\mathscr{S}$ to be the class of closed continuous embeddings and that of continuous onto-mappings. The one point space is a singleton in $\mathscr{C}_M$, so the dual condition (A_1^*) of (A_1) is satisfied. If $\{A_i\}_{i\in I}$ is a system of closed subspaces of a space A, then the closure $\overline{\bigcup_{i\in I} A_i}$ of the union of the subspaces will be, clearly, the categorical union of the subspaces A_i, $i\in I$. Hence also (A_2^*) is fulfilled in $\mathscr{C}_M$.

To show the validity of (A_3^*), let us remark that in the category of topological spaces the coproduct is precisely the disjoint union of the spaces. We shall show that the disjoint union of complete metric spaces is again a complete metric space. By [2] Theorem 4. 2. 1 this disjoint union is a metric space. Consider a Cauchy sequence $\{x_n\}$ in the disjoint union $A=\bigoplus_{i\in I} A_i$. Now to any $\varepsilon>0$ there exists a natural number N such that $\varrho(x_n, x_m)<\varepsilon$ holds for every $n, m\geqq N$. This is possible only if x_n and x_m belongs to the same space A_i for a fixed $i\in I$. Since A_i is complete, so the sequence $\{x_n\}$ is convergent in A_i as well as in A.

(A_4^*) $\mathscr{C}_M$ admits inverse limit of countable inverse systems. Since the Cartesian product of a countable number of complete metric spaces is again such a space ([2] Theorem 4. 3. 7), so taking into account that the inverse limit is a closed subspace of the Cartesian product, the validity of (A_v^*) is obvious.

(C*) Let us consider an inverse system $\{A_i, i=1, 2, \ldots\}$ of quotient spaces $A_i\in\mathscr{C}_M$ of a space $A\in\mathscr{C}_M$. First of all we shall show that the canonical map $\gamma\colon A\to\varprojlim\{A_i\}$ is an onto-mapping. By (B_4) γ can be factored as $\gamma=\psi\varphi$ with $\psi\in\mathscr{I}$ and $\varphi\in\mathscr{S}$ such that diagram *(16)* is commutative for $i=1, 2, \ldots$. Consider an arbitrary element $a\in\varprojlim\{A_i\}$. Obviously a has the form $(\ldots, a_j, \ldots, a_i, \ldots)$ with $\pi_j^i a_i=a_j$. The inverse image $F_i=(\pi_i\psi)^{-1}(a_i)$ is a closed subset of C,

$$\begin{tikzcd}
 & C \arrow[d, "\psi"] \\
A \arrow[ur, "\varphi"] \arrow[r, "\gamma"] \arrow[dr, "\alpha_i"'] & \varprojlim\{A_i\} \arrow[d, "\pi_i"] \\
 & A_i
\end{tikzcd}$$

(16)

moreover $F_1 \supset F_2 \supset \dots$ holds. If $\delta(F_i)$ denotes the diameter of F_i, then by [2] Theorem 4. 2. 2 we have

$$\delta(F_i) \leqq \delta(\pi_i^{-1}(a_i)) \leqq \frac{1}{2^i}.$$

Thus $\lim_{i \to \infty} \delta(F_i) = 0$, and so the completeness of C implies that the intersection $\bigcap_{i=1}^{\infty} F_i$ is not empty. For $b \in \bigcup_{i=1}^{\infty} F_i$ we have, clearly, $\psi(b) = a$, and so ψ is a surjection too. Hence γ is indeed an onto-mapping.

If $\delta : A \to D$ is a surjection such that diagram *(17)* is commutative for $i = 1, 2, \dots$, then diagram *(18)* is also commutative.

$$\begin{tikzcd} & & D \arrow[dl] \arrow[dd, "\delta_j"] \\ A \arrow[urr, "\delta"] \arrow[r, "\alpha_i"] \arrow[drr, "\alpha_j"'] & A_i \arrow[dr, "\pi^i_j"] & \\ & & A_j \end{tikzcd}$$

(17)

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\alpha_i"'] & D \arrow[d, "\delta_i"] \\ & A_i \end{tikzcd}$$

(18)

Hence by the definition of the inverse limit, for the canonical map $\delta' : D \to \varprojlim \{A_i\}$ we get $\delta'\delta = \gamma$. Now $\gamma \in \mathcal{S}$ implies $\delta' \in \mathcal{S}$, and so $\varprojlim \{A_i\} = \bigcup_{i=1}^{\infty}{}^{*} A_i$ is proved. Thus $\mathcal{C}_M$ fulfils condition (C^*).

A reformulation of the dual statement of the Theorem is

Theorem.* *Let* **P** *be a topological property of complete metric spaces such that it is inherited for closed subspaces, and it is preserved to the closure of the union of countable many subspaces belonging to* **P**. *Then property* **P** *is preserved by forming inverse limit of countable inverse systems of complete metric spaces.*

To motivate Theorem*, let us choose property **P** as follows:

a) **P** means the *Lindelöf property*;

b) **P** means that *the space A has weight* $w(A) \leqq \mathfrak{m} (\geqq \aleph_0)$.

Let us recall that in $\mathcal{C}_M$ the Lindelöf property is equivalent to the separability, and $w(A) \leqq \mathfrak{m}$ means exactly that A contains a dense subset of cardinality $\leqq \mathfrak{m}$ (cf. [2], Chapter 4).

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Bibliographie

E. Artin, Galoische Theorie (Mathematisch-Naturwissenschaftliche Bibliothek, 28), iv + 86 pages, B. G. Teubner Verlagsgesellschaft, Leipzig, 1959. — DM 5.30

The book is a revised German translation by V. ZIEGLER of the English original (*Galois theory*, 2nd ed., Notre Dame University, Indiana, USA, 1948). Its aim is to make one familiar with the methods and problems of Galois theory. As no preliminary knowledge is presupposed from algebra, it appeals to every moderately advanced student with a sufficiently inquisitive mind.

The work is divided into three parts. The first one studies the basic concepts of linear algebra such as vector spaces over skew fields, systems of homogeneous linear equations, ranks of matrices, and determinants. The second part considers commutative fields and their algebraic extensions, among them separable and normal ones. As applications of the theory given here, the last part studies some special questions. After an account of solvable groups, the familiar criterion for solvability by radicals of an algebraic equation is presented. An equation of degree 5 is given that is not solvable in this way. Finally, construction by ruler and compass is discussed.

Simultaneously with the translation, some major improvements were carried out by the author in this editions such as: the proof of the fundamental theorem of Galois theory is rendered more coherent; a proof, based on E. LANDAU's ideas, of the irreducibility of the cyclotomic polynomials is included in the section on roots of unity; and the last part of the book has been rewritten entirely.

The careful selection and the clear presentation of the material are among the greatest virtues of this book.

A. Máté (Szeged)

G. Asser, Einführung in die mathematische Logik. I. Aussagenkalkül (Mathematisch-Naturwissenschaftliche Bibliothek, 18), VI + 184 pages, B. G. Teubner Verlagsgesellschaft, Leipzig, 1959. — DM 11.25

This is the first book of a planned three-volume introduction into mathematical logics. This volume deals with propositional calculus; and, as is claimed in the preface, the second one is going to discuss first-order predicate calculus; finally, the third one is planned about higher order calculi. The work intends to remain within the bounds of traditional two-valued logics, and no discussion on intuitionism, applications to foundations of mathematics, or philosophical problems, etc., will be included.

Within these limits, the scope of the present volume is fairly comprehensive, though, of course, no book of this size can aim at completeness. It gives a detailed treatment of propositional expressions; a large number of tautologies is listed; a long discussion on the theory of propositional normal forms is presented; deducibility in propositional calculus is dealt with, and its completeness is proved; axiomatizability, consistency, and completeness are studied; independence, propositional matrices, and deductively closed sets are discussed; representability of functions in various extensions (via propositional matrices) of the propositional calculus is dealt with; a syntactic characterization, based on ideas going back to G. GENTZEN, is given of expressions of the classical propositional calculus; the book ends with an account of the generalized notion of calculi.

A. Máté (Szeged)

Leonard M. Blumenthal and Karl Menger, Studies in geometry, XVI+512 pages, San Francisco, W. H. Freeman and Company, 1970.

This book represents on a graduate course level a wide variety of research subjects in modern geometry initiated by works of L. M. BLUMENTHAL and KARL MENGER.

Part 1, written by BLUMENTHAL treats geometrical aspects of lattice theory and of Boolean algebras. After a review of basic concepts and facts normed lattices are introduced and questions concerning their metric geometry discussed. In Boolean algebras a generalized distance of two elements, which is again an element of the algebra, is defined and shown to possess the basic properties of ordinary distance. The Boolean metric spaces thus obtained give rise to a generalized metric geometry on their own.

In Part 2 an exposition of the foundations of projective geometry is given by MENGER, based on the concept of projective structures introduced by him in 1932. This approach having remarkable advantages is worked out for 3 dimensions including results on Arguesian and Pappus planes and Dandelin spaces.

Part 3, written by BLUMENTHAL gives a survey of metric geometry. Basic results concerning metric segments and lines of normed linear spaces are presented first to be applied to various metric characterizations of Banach and euclidean spaces. Some sections treat topics from the metric theory of curves pertaining generally to an at most 3-dimensional situation. This part ends with a concise presentation of the metrization of the Gauss curvature by A. WALD and of related results due to W. A. KIRK.

In Part 4 MENGER gives with characteristic lucidity an expositions of topological curve theory which also embodies results following the edition of his famous *Kurventheorie*.

The whole book is a very suggestive new illustration to the fruitfulness of MENGER's well-known objective to bring traditional topics into the scope of modern mathematics.

J. Szenthe (Szeged)

Functional Analysis and Related Fields, Proceedings of Conference in honor of Professor Marshall Stone, held at the University of Chicago, May 1968. Edited by F. E. BROWDER, 1 portrait VIII+241 pages, Berlin—Heidelberg—New York; Springer-Verlag, 1970. — DM 58.—

The volume contains a number of important contributions. Contents: F. E. BROWDER: Nonlinear eigenvalue problems and group invariance; S. S. CHERN, M. DO CARMO, and S. KOBAYASHI: Minimal submanifolds of a sphere with second fundamental form of constant length; HARISH-CHANDRA: Eisenstein series over finite fields; E. HEWITT: $\mathscr{L}_p$ transforms of compact groups; T. KATO and S. T. KURODA: Theory of simple scattering and eigenfunction expansions; G. W. MACKEY: Induced representations of locally compact groups and applications; L. NACHBIN: Convolution operators in spaces of nuclearly entire functions on a Banach space; E. NELSON: Operants: A functional calculus for non-commuting operators; I. SEGAL: Local non-linear functions of quantum fields; A. WEIL: On the analogue of the modular group in characteristic p; A. ZYGMUND: A theorem on the formal multiplication of trigonometric series; S. MACLANE: The influence of M. H. Stone on the origins of category theory; Remarks of Professor STONE.

The topic of most of these papers is intimately related to some aspects of the oeuvre of M. H. STONE (spectral theory, operators, Boolean algebras, general topology, Weierstrass—Stone theorem, harmonic analysis, etc.). Stone's theorem on one parameter groups of unitary operators, and the Stone — von Neumann uniqueness theorem for the Weyl commutation relations in quantum mechanics are, for example, basic for the "imprimitivity theorem" of Mackey and for its consequences, an excellent survey of which is contained in the paper of MACKEY. The topics of the papers of KATO—KURODA, and SEGAL are also closely related to that theorem of Stone.

Less known is the influence of the work of Stone on the origins of Category Theory. This influence is discussed in the very interesting paper by MACLANE. In particular, the influence of Stone's work on adjoint linear operators and on the representation of Boolean rings is considered. In his Remarks, STONE makes a few comments by way of response to some historical questions raised by MACLANE about the concept of adjoint functors or operators, and also discusses some general aspects of mathematical research. He points to "the need for offering our future research mathematicians a broad preparation that will enable them to cope successfully with the increasingly complex interconnections that bind mathematics into a single whole."

Béla Sz.-Nagy (Szeged)

J. C. Burkill and H. Burkill, A Second Course in Mathematical Analysis, VII+526 pages, Cambridge University Press, 1970.

This book is a continuation of "A First Course in Mathematical Analysis" by J. C. BURKILL, accordingly the abovementioned First Course is the best foundation for this second Course. The book is intended for mathematics students who are familiar with the concept of a limit and its applications to infinite series and to the differential and integral calculus.

The subject is presented in the more abstract setting of metric spaces. The first nine chapters concentrate on general analysis and real functions, and the last five on complex functions. The chapter headings indicate the scope of the book in more detail. They are: 1. Sets and Functions, 2. Metric Spaces, 3. Continuous Functions on Metric Spaces, 4. Limits in the Space R^1 and Z, 5. Uniform Convergence, 6. Integration (The Riemann and Riemann—Stieltjes integrals are treated but the Lebesgue integral is left out), 7. Functions from R^m to R^n, 8. Integrals in R^n, 9. Fourier Series, 10. Complex Function Theory, 11. Complex Integrals. Cauchy's Theorem, 12. Expansions. Singularities. Residues, 13. General Theorems. Analytic Functions, 14. Applications to Special Functions.

The book contains nearly seven hundred exercises and problems with their solutions, and, furthermore the authors give some illustrations of the numerous definitions.

At the end of each chapter there are given notes, some historical ones and others indicating further developments. A list of references and a detailed index are added.

The book is lucidly arranged and the exposition is clear.

L. Leindler (Szeged)

Herbert Busemann, Recent synthetic differential geometry, VI+110 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970.

The author's approach to global problems of intrinsic differential geometry presented in his book "*The Geometry of Geodesics*" has proved to be superior to the standard ones in several problems especially in those which concern Finsler geometry. Since the issue of this book in 1955 various results have been achieved on subjects already treated or but proposed in it, and now the author gives a systematic account of these recent results.

In Chapter I questions concerning the foundations of the theory are treated. Since the basic role of the HOPF—RINOW theorem in global differential geometry is in striking contrast with its usually rather loose presentation, a thorough analysis as to the assumptions of this theorem is given first. Next the problem of the topological structure of the so-called G-spaces is considered. These G-spaces are metric spaces satisfying a system of axioms motivated by differential geometry and are the basic concepts in the author's theory. In spite of their geometric versatility the topological structure of these spaces does not admit an easy description. By a modification of the theory of the r-spaces of A. KOSINSKI the author proves that finite dimensional G-spaces possess the property of domain invariance and that small spheres of a G-space are not contractible. The proof of the theorem of B. KRAKUS that every 3-dimensional G-space is a topological manifold is given as well. An earlier result of the author is also presented, stating that a continuously differentiable and regular G-space is a topological manifold, where, of course, continuous differentiability is meant in a metric sense.

Desarguesian G-spaces form the subject of Chapter II. Here the concept of similarity, introduced earlier by the author, is studied first. Then a solution of a problem on the imbedding of desarguesian G-spaces is presented together with results of the author on those r-dimensional areas in n-dimensional affine space for which the r-flats minimize the area. A characterization of HILBERT's and MINKOWSKI's geometries among all straight desargueasian spaces in given by generalizing a theorem of L. BERWALD on desarguesian spaces with constant Finsler curvature. In this context the result of P. KELLY and E. G. STRAUSS on the characterization of hyperbolic geometry among HILBERT's geometries by a metric curvature concept is mentioned.

Results on length preserving maps are presented in Chapter III. Some basic observations on shrinkages, equilong maps and local isometries are made first. Then several conditions are presented under which a G-space cannot have proper local isometries. These conditions have been given by the author, W. A. KIRK, and the reviewer. For elementary spaces all those equilong maps are constructed which are locally finite in the sense that the covering by the closures of their regions of injectivity is locally finite.

Chapter IV contains results on the behaviour of geodesics. First a theorem of the author is presented which he managed to prove by applying a method due to V. A. EFREMOVIČ and E. S. TIHOMIROVA. By means of this theorem it is then shown that on a closed hyperbolic space form with an intrinsic, but not necessarily symmetric distance there is a class of geodesics and half geodesics which behave uniformly like the hyperbolic geodesics and half geodesics. The facts which make axes of motions interesting have been pointed out by the author and F. P. PEDERSEN. These facts are reproduced here and some remarks are made how results of H. HEDLUND, M. MORSE and G. A. BLISS can be extended and proved in a more elegant manner. Inverse problems of the calculus of variations are considered next and results of the author, H. SALZMANN and L. A. SKORNYAKOV are presented concerning the so-called collineation groups, especially the 1-dimensional and the discrete ones. Due to the fundamental role of the GAUSS-BONNET theorem in the discussion of the behaviour of geodesics in the large on 2-dimensional Reimannian surfaces the question concerning the generalization of this theorem to FINSLER surfaces is essential. Since numerous different generalizations are known it is shown that a generalization comprising the two central features of the theorem does not exist. The result of E. KANN on the generalization of BONNET's theorem concerning the diameter of Riemannian manifolds to G-surfaces is mentioned with some interesting remarks and an observation of E. M. ZAUSTINSKY is presented concerning the divergense property of geodesics on G-surfaces. A brief report is given on results of G. M. LEWIS on conjugacy to points at infinity, a problem taken up initially by J. NASU..

Results on motions are presented in Chapter V. Spaces with a finite or 1-parameter group of motions are considered first and spaces with a group of motions transitive on the set of geodesics second.

Chapter VI contains observations on the contents of the theory and the methods applied in it. A brief outline of the development of the FINSLER geometry is also given with some very interesting criticism on trends of investigations. Several objectives for further research are mentioned.

By commenting on new results and presenting them in the context of the theory as a whole the author has given an authentic review of an advancing trend of investigations following the issue of "*The Geometry of Geodesics*".

J. Szenthe (Szeged)

O. Bottema, R. Ž. Djordjević, R. P. Janić, D. S. Mitrinović, P. M. Vasić, Geometric inequalities, 151 pages, Groningen, Wolters-Noordhoff Publ. Co., 1969.

Die geometrischen Ungleichungen sind so alt, wie die Geometrie selbst, zahlreiche solche Ungleichungen stammen aber aus den letzten zweihundert Jahren, ja einige sogar aus den kürzlich vergangenen Jahrzehnten. Es handelt sich insbesondere um Ungleichungen für die Seiten eines Dreiecks bzw. von zwei Dreiecken, über spezielle Dreiecke und über Ungleichungen von Vierecken.

Die einfacheren Ungleichugen werden im Buch bewiesen, über die schwereren aber geben uns reichliche Hinweisungen und Anführungen aus der Literatur Auskunft. Die Verfasser haben auch die neueste einschlägige Literatur benutzt.

J. Berkes (Szeged)

K. Chandrasekharan, Arithmetical functions (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 167), XI+231 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1970. — DM 58. —

As is claimed in the introduction, this book can be looked upon as a sequel to the author's *Introduction to analytic number theory,* Berlin—Heidelberg—New York, Springer-Verlag, 1968. (For a review see these *Acta*, **31** (1970), p. 185.) Its title might be slightly misleading: namely, it gives account on an advanced level of problems associated with the distribution of primes, the partition function, and the divisor function. On the part of the reader a moderate acquaintance with the methods of analytical number theory is only assumed. This can be acquired from the work just cited.

The style of the book is clear, the proofs are carefully arranged. Nevertheless, at some places one might wonder whether a little more intuitive description of the ideas behind the proofs would not help the reader to struggle himself through the heaps of symbols that are inevitable in such a context.

To give a more precise description of the contents of the book, here are the chapter headings: I. The prime number theorem and Selberg's method, II. The zeta-function of Riemann, III. Littlewood's theorem and Weyl's method. IV. Vinogradov's method, V. Theorems of Hoheisel and Ingham, VI. Dirichlet's L-functions and Siegel's theorem, VII. Theorems of Hardy—Ramanujan and of Rademacher on the partition function, VIII. Dirichlet's divisor problem. Each chapter ends with notes containing additional information related to the subject; thus the reader is spared the diversion of attention that could have been caused by giving this information in footnotes or interjected remarks. The book ends with a small *List of books* related to the topics discussed and a *Subject index.*

As the above description may show, the book is on the whole really good. The printer's contribution to its value should certainly be mentioned: the format is very pleasing. Considering the large number of formulas in the text, this must have needed a very meticulous effort.

A. Máté (Szeged)

A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Vol. II, XV+350 pages, Providence, Rhode Island, American Mathematical Society, 1967.

People interested in semigroup theory have been looking forward to the publication of this second part of the monograph of CLIFFORD and PRESTON. The work thus completed is undoubtedly the best one in the field. Though Volume II does not attain the classical compactness of Volume I, this being almost impossible because of the present unaccomplished state of investigations, it gives a clear outline of the trends in the development of the theory. Its best parts, as §§ 6.1—6.4, 9.4—9.5, Chapter 12, and others, will make a pleasure to those liking mathematics not only as science but as an art as well.

The main deficiency of the book is that it makes sometimes the impression of a selection of papers; the authors failed in fusing their material into an organic whole. A simple example: on pp. 108 and 109 there are two constructions. Both amount to an extension of a semigroup by the bicyclic semigroup and yet even the notations used in the two cases are essentially different.

Here is a short résumé of the contents of the book:

Chapter 6 deals with minimal ideals and minimal conditions. Its main part is constituted by building up ŠT. SCHWARZ's theory of the right (and left) socles of a semigroup, and culminates in the authors' theorem about the structure of the union of the left and right socles. The last section contains some results concerning semigroups with minimal condition on left (or right) ideals.

In Chapter 7 the authors give a survey of results in the theory of inverse semigroups, as those of B. M. ŠAIN about the representations of inverse semigroups by one-to-one partial transformations, the characterization of congruences in inverse semigroups by their classes containing idempotents, and some special questions concerning such congruences (idempotent separating congruences etc.).

Chapter 8 continues the study of simple semigroups, considered already in Chapters 2 and 3, Volume I, and it contains some problems of embedding semigroups in simple semigroups of different types, too. The discussion ends with PRESTON's theorem claiming that any semigroup can be embedded in a bisimple semigroup with identity.

After some preliminary information on free semigroups, Chapter 9 contains RÉDEI's theorem on finitely generated commutative semigroups (all semigroups of this kind are finitely presented), and HOWIE's results concerning the embeddability of a set of semigroups S_i with a common subsemigroup $\cap S_i = U$ into a larger semigroup. As for the first one, P. FREYD has since then found an amazingly simple proof (*Proc. Amer. Math. Soc.*, **19** (1968), 1003). In the last section the authors discuss the method of constructing the minimal cancellative congruence containing a given relation, on a semigroup S.

The first part of Chapter 10 presents the basic results of the theory of DUBREIL's principal one-sided congruences and those of CROISOT in the field of two-sided congruences. Then there follows a specialization for semigroups of GOLDIE's generalized Jordan-Hölder theorem, the analysis of the congruences of completely 0-simple semigroups, and the description of the congruences of the full transformation semigroup of a set X, due to MALCEV.

The next chapter is devoted to the theory of semigroup representations by transformations of a set.

The last chapter seems to be the authors' gift to the reader: it contains, in a highly delectable presentation, the results of LAMBEK and MALCEV concerning embeddability in a group.

Of course, one can always find interesting subjects omitted from a monograph, and there is little use in listing them; nevertheless, it seems to the reviewer that a chapter devoted to different methods of semigroup constructions could contribute not only to the completeness but also to the coherency of the work.

G. Pollák (Szeged)

F. M. Hall, An introduction to abstract algebra, Vol. 2, XII+388 pages, Cambridge University Press, 1969. — £ 3.50

This book, formally the second part of a two-volume textbook for abstract algebra, is a self-contained introduction to the theory of the most important algebraic structures, as groups, rings, fields, vector spaces, and Boolean algebras. It can be used by undergraduate students and their teachers as an auxiliary material, and it is also excellent for self-education.

An informal style without loss of exactness and a didactical construction make this volume an excellent explanation of its subject-matter. The author avoids successfully the customary Scylla and Charybdis in this genre: tedious sophistication and disproportion in the selection of topics. We mention also the well-chosen exercises (there are about seven hundred of them).

B. Csákány (Szeged)

F. Hausdorff, Nachgelassene Schriften. Bände I, II: Studien und Referate. Herausgegeben von G. Bergmann. Bd. I: XXI+538 pages, Bd. II: IX+570 pages, Stuttgart, B. G. Teubner, 1969. — DM 122. —

These two illustrious volumes contain a large number of yet unpublished writings of F. HAUSDORFF (1868—1942) in form of facsimile reproduction of the original hand-written manuscripts. The writings are selected from studies and reviews originating from the interval March 1934 through March 1938. In about two years, further two volumes are planned, selecting from writings prepared between March 1938 and January 1942. Afterwards, the publication of earlier wirtings may also be contemplated.

It is simply astonishing, and rather unfortunate, that so great a number of Hausdorff's writings remained unpublished. The reason for this is perhaps partly found in Hausdorff's personality itself. But it must not be forgotten that the circumstances in Germany were then unfavourable to the publication of these works. Indeed, in the last years of his life, Hausdorff was actually barred from publishing.

The value of these writings is enormous. A review by Hausdorff has quite exceptional qualities. It is very detailed and thoroughgoing; and if it does not actually pinpoint an unaccuracy, then we may take it as a guarantee of the faultlessness of the reviewed work. As for the studies, they are written with crystal-clear logic and precision. Most of them centre around topics in topology and set theory. It is certainly great loss that these writings have not been available earlier. But even now, they should be of great importance, not only to those turning to Hausdorff's activity with an historical interest, but also to those interested in the subjects discussed; even though the handwritten format seems to favour the former ones.

The handwriting of Hausdorff is quite well legible and the technical quality of the reproduction is excellent.

A. Máté (Szeged)

C. A. Hayes and C. Y. Pauc, Derivation and Martingales (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 49), VII+203 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1970.

Today we are witnessing a new revival of the theory of derivation and, in a close connection with this, of the modern theory of martingales. But there is no book, to the knowledge of the reviewer, to collect and systematize the essential results of the subject. This is the first attempt, and an excellent one at that, to survey the new methods and results of the modern theory of derivation in a single volume.

The work is relatively self-contained, and all of it can be read by anyone with a preliminary knowledge in the elements of set theory, topology, and measure theory. Most of the notions and results needed from these fields are presented and explained in the text, though in a way presupposing a certain general mathematical maturity of the reader.

The book consists of two parts, in six and four chapters, respectively, a *Complements*, and a *Subject Index*. Each part has its own bibliography, which is short but to the point.

Part I is devoted to the pointwise derivation of scalar set functions employing, in general, abstract derivation bases or blankets, due to R. DE POSSEL and A. P. MORSE, respectively. The principal tool is a Vitali property, of strong or weak type, whose precise form depends on the derivation property studied. The converse problem is also investigated: what covering properties can be deduced from derivation properties of σ-additive set functions.

The "halo" properties furnish the foundation for many of the modern results to establish a Vitali property, or sometimes to produce directly a derivation property. A few sections are concerned with the abstract version of the strong Vitali theorem, modelled after those given by BANACH or CARATHÉODORY. The main results presented are the theorem of JESSEN—MARCINKIEWICZ—ZYGMUND, valid in n-dimensional Euclidean spaces R^n with the interval basis, and the theorem of MORSE on the universal derivability of star blankets. In the context of the former one, the maximal theorem of HARDY—LITTLEWOOD is proved for R and R^2.

Part II begins by studying the notion of increasing stochastic bases with directed index sets on which premartingales, semimartingales and martingales are defined. Convergence theorems, due largely to K. KRICKEBERG, are treated in great detail using various types of convergence: stochastic, in the mean, in L_p-spaces, in Orlicz spaces, and with respect to the order relation. To each theorem for martingales and semimartingales there corresponds a theorem in the atomic case in the theory of cell functions, where cells can be construed as generalized intervals. The derivates concerned are global.

Finally, in a separate chapter, concepts are reintroduced and results on pointwise convergence and on point derivates are deduced from results obtained in the earlier chapters of Part II, under supplementary assumptions. To mention one example, the Radon-Nikodym integrand is defined as a derivative.

The *Complements* consists of such sketches related to topics in Part I and II as derivation of vector-valued integrals, global derivatives in locally compact topological groups, vector-valued martingales and derivation, etc.

The presentation of the book is concise but always clear and well-readable. It should have a great appeal to the students of graduate courses as well as to the mature mathematicians interested in the modern theory of derivation.

F. Móricz (Szeged)

H. P. Künzi and W. Oettli, Nichtlineare Optimierung: Neuere Verfahren. Bibliographie (Lecture Notes in Operations Research and Mathematical Systems, Vol. 16) 180 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1969. — DM 12, —

Das Buch knüpft sich in mehrerer Hinsicht an das im Jahre 1962 erschienene Werk von H. P. KÜNZI und W. KRELLE, *Nichtlineare Programmierung*. Sie ist auch insofern als eine Fortsetzung der erwähnten Arbeit zu betrachten, daß sie nur die Bearbeitung der Forschungsergebnisse enthält, die nach 1960 publiziert wurden.

Die Verfasser betonen, daß sie bei der Ausarbeitung des Themas die Vollständigkeit nicht erzielt haben, es ist aber ohne Zweifel, daß hier die nichtlineare Programmierung und hauptsächlich die wichtigsten neuen Methoden der konvexen Programmierung Platz bekamen. Man betrachtet insbesondere die folgenden Methoden:

Das Schnittebenenverfahren von Kelley; die tangentielle Approximationsmethode von Hartley und Hocking; die modifizierten Schnittebenenverfahren von Kleibohm, Veinott und Zoutendijk; MAP (Method of Approximation Programming) von Griffith und Stewart; die reduzierte Gradientenmethode; die Methode der Penalty-Funktionen; SUMT (Sequential Unconstrained Minimization Technique) von Fiacco und McCormick; die Zentrumsmethode von Huard; und ein Verfahren der zulässigen Richtungen.

Die Beschreibung der Methoden erstreckt sich auf die theoretischen Grundlagen, auf die Gedankenfolge, die zur Methode führt, es geht auch auf ihre Anwendungsweise, auf deren Möglichkeiten und Grenzen ein. Es handelt sich hier auch um Konvergenzprobleme und natürlich um alle weiteren Probleme, die mit der gegebenen Methode in Verbindung stehen. Die Behandlung geht — sehr richtig — auf Einzelheiten nicht ein, zahlreiche Behauptungen sind ohne Beweis angegeben. (Die Beweise sind auf Grund des angegebenen Literaturverzeichnisses vorzufinden.)

Mehr als die Hälfte des Umfangs nimmt eine Bibliographie ein. Hier wurden die Werke aufgeführt, die zu dem Themenkreis der nichtlinearen Programmierung gehören oder mit ihm in enger Verbindung sind, dessen theoretische Grundlage oder seine Anwendung bilden. (Die Arbeiten, die zu dem Kreis der dynamischen, stochastischen und ganzzahligen Programmierung gehören, sind nicht einbezogen.)

Das Buch gibt ein getreues Bild über den heutigen Stand der nichtlinearen Programmierung.

L. Megyesi (Szeged)

Gilbert Helmberg, Introduction to Spectral Theory in Hilbert space, XIII+346 pages, Amsterdam—London, North-Holland Publ. Co., 1969. — Hfl. 60, —

The purpose of the book is to be an introduction to the subject and no part of it is claimed to be original. However, the author expresses "the hope that among the many different people interested in this subject there might be some who find this presentation particularly suited to their personal taste."

Chapters: The concept of Hilbert space. — Specific geometry of Hilbert space. — Bounded linear operators. — General theory of linear operators. — Spectral analysis of compact linear operators. — Spectral analysis of bounded linear operators. — Spectral analysis of unbounded selfadjoint operators.

There is an Appendix recalling some pertaining results of the theory of Real Functions.

B. Sz-Nagy (Szeged)

P. Lorenzen, Formale Logik (Sammlung Göschen, Bd. 1176/1176a), second corrected edition, 165 pages, Walter de Guyter & Co., Berlin, 1962. — DM 5,80

A review of the first edition appeared in these *Acta*, **20** (1959), p. 219, where a strong criticism was directed against the book. In what this review was undoubtedly right was the pinpointing of some disturbing misprints. It seems that the editor laid a great care on eliminating them in this second, corrected edition. Also, true enough, as the cited review observed, the slightly complicated notational framework of the book is not to be praised. But the criticism seems certainly unfair in bypassing the virtues of this book.

The main virtue of this small paperback is that it is a pleasure to read it. The style is very lively, full of background explanations. Thereby it is inevitable that the author puts forward his own views on the foundations of logics, and so the treatment is influenced by the author's "operative views" on mathematics (cf. P. LORENZEN, *Einführung in die operative Logik und Mathematik,* 2nd edition (Berlin — Heidelberg — New York, 1969); for a review see these *Acta,* **30** (1969), p. 329). Since this view is not shared by most mathematicians, this feature certainly makes the book less attractive to those starting to learn mathematical logic, but, on the other hand, it gives a special flavour to the book.

The material comprises accounts of syllogisms, of a logic of junctors, of an effective logic of junctors, of a logic of quantifiers, and of logic of equality. Under these titles are also discussed such classical results as the consistence and completeness of the propositional calculus, completeness of the predicate calculus, Church's theorem, etc.

A. Máté (Szeged)

I. J. Maddox, Elements of Functional Analysis, X+208 pages, Cambridge University Press, 1970. — 50s.

As the author states in the Preface, in his view "the field of elementary functional analysis is the ideal place in which to learn some abstract structural mathematics and to develop analytical technique". He presents now a book which provides an introductory, though non-trivial, course on functional analysis which can be followed by every student who has completed basic courses on real and complex variable theory. Most of the examples which are chosen to motivate the basic concepts or to illustrate the strength of the results achieved, involve sequence spaces rather than integration spaces; thus Lebesgue integral theory is not an absolutely necessary prerequisite (however, completeness of the L_p spaces is proved by referring to the relevant theorems on the interchange of limit and integration).

Chapter titles (and some key words): 1. Basic set theory and analysis. (Zorn's lemma.) — 2. Metric and topological spaces. (Category and uniform boundedness.) — 3. Linear and linear metric spaces. (Hamel "base" and Schauder "basis".) — 4. Normed linear spaces. (Open mapping, closed graph, and Hahn-Banach theorems.) — 5. Banach algebras. (Gelfand representation theorem.) — 6. Hilbert space. (Orthonormal sets.) — 7. Matrix transformations in sequence spaces. (Summability. Tauberian theorems.)

There is a great number of exercises, and the last chapter (which concerns an area of special interest to the author) ends with problems for further study, some of them quite difficult.

Operators and spectral theory are barely touched. But within its limits chosen, the book is a useful introduction to the theory, written with much didactical care.

Béla Sz.-Nagy (Szeged)

Jaques L. Mercier, An introduction to tensor calculus, VIII+152 pages, Wolters-Noordhoff Publishing, Groningen, 1971.

The wide variety of topics exposed nowadays generally in tensor form in the technical and scientific literature requires a working knowledge of tensor calculus from the interested reader. This book is intended to help students and engineers to provide themselves with such a knowledge.

Part I is an introduction to the invariant formulation of tensor calculus. The subjects which are exposed here range from the fundamental concepts to such ones as Riemannian tensors. In Part II covariant differentiation is introduced and developed to an extent required by its applications in engineering science.

The topics treated in this book are of course very much the same as in the numerous others of its kind. Yet, by his exceptional care to attain a reasonable maximum of mathematical rigour and by his personal skill in composing an ideal blend of illustrative examples and exercises the author succeeded in writing a book which, owing to these distinguishing qualities, would surely be welcome by a wide class of readers.

J. Szenthe (Szeged)

Jean Pierre Serre, Abelian *l*-adic representations and elliptic curves, New York—Amsterdam, W. A. Benjamin, Inc., 1968.

The book, written in collaboration with JOHN LABUTE and WILLEM KUYK, reproduces with a few complements the lectures of the author at McGill University, Montreal, in 1967. The *l*-adic representations treated here have been introduced by Y. TANIYAMA in 1957 and are the algebraic analogue of the locally constant sheaves of topology.

In Chapter I the definition and some examples of *l*-adic representations are given first. Then assuming that the ground field is a number field rational *l*-adic representations are considered. The attaching of *L*-functions to rational *l*-adic representations is discussed too.

Chapter II contains the construction of some abelian *l*-adic representations of a number field, which is originally due to G. SHIMURA, Y. TANIYAMA and A. WEIL, and is given here with some modifications.

In Chapter III the question is considered whether an abelian *l*-adic representation of a number field can be obtained by the method of the preceding chapter, and in this respect a necessary and sufficient condition is given. The problem whether any abelian rational semi-simple *l*-adic representation of a number field is ipso facto locally algebraic is also taken up and proved for the case when the field is a composite of quadratic fields.

Chapter IV is concerned with the *l*-adic representation defined by an elliptic curve. Its aim is to determine, as precisely as possible, the image of the Galois group, or at least the corresponding Lie algebra.

A considerable number of instructive exercises and abundant motivation by remarks and references serve to make this book a very readable exposition.

J. Szenthe (Szeged)

C. A. Rogers, Hausdorff measures, viii + + 179 pages, London, Cambridge University Press, 1971. — £ 3.8.

The theory of Hausdorff measures was initiated by C. CARATHÉODORY in 1914 when he studied linear and *p*-dimensional measures in *n*-dimensional Euclidean space, thus giving a general solution to the problem of measuring surfaces. This theory was further developed by F. HAUSDORFF; as an illustration of his results he showed how to assign a positive finite measure to Cantor's ternary set in a natural way. Since then the progress in the field has been enormous, largely due to the work of A. S. BESICOVITCH and his students. Despite this progress, until now there has been no book discussing the fundaments of the subject.

A book of this size cannot serve as general reference on a subject with such diversified applications as the theory of Hausdorff measures, but this one does achieve a lot. It outlines the main core of the theory; it sets down a standard terminology and restates many results scattered in the mathematical literature or known as "folklore", in a sufficiently general from. These features make the book indispensable for research mathematicians in measure theory; but the simple style makes it also very attractive to students.

Measures have two, more or less distinguishable, major roles in mathematics. First, they can serve for sizing sets, and, second, they can be used to define integrals. In the present work, naturally, the first aspect prevails. We are going to give a closer description of the contents:

The book is divided into three parts. The first one studies the general aspects of measure theory. The author departs from the standard terminology by calling measure what is usually referred to as outer measure (we shall do the same below). The construction from pre-measures and the properties of measures are discussed, with a special stress on measures in topological and in metric spaces, and on non-σ-finite measures in general. The chapter ends with an account of the Souslin operation. The second chapter deals with the more general aspects of Hausdorff measures. After their definiton, their behaviour with respect to mappings and their use for measuring surface areas are considered. Existence and comparison theorems are studied extensively. Accounts of Souslin sets, of consequences of the increasing sets lemma, and of comparable net measures follow. Finally, a special section is devoted to the investigation of non-σ-finite sets. The topic of the last chapter is the applications. The first section here is a general survey of them, mentioning only the most important ones. The first of the applications given in detail is an account of JARNÍK's pioneering investigations concerning sets of real numbers defined in terms of their expansions into continued fractions. The second one describes a part of the study of S. J. TAYLOR and the author on additive set functions in Euclidean space (this is the only part of the book where integration is also considered). The book ends with an extensive bibliography and an index.

A. Máté (Szeged)

I. Singer, Bases in Banach Spaces. I. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band 154), VIII + + 668 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970.

The concept of basis is a fundamental tool to investigate the structure of various abstract spaces in functional analysis such as Banach spaces, F-spaces, topological linear spaces, etc. Although there are at present possibly more than a thousand publications in existence on the theory of bases, there has to date been a serious gap in the textbook literature. The recent appearance of two books helps to fill this gap, and it will most certainly be a major stimulant to the development of the subject.

The books are, in order of appearance, *Introduction to the theory of bases,* by J. T. MARTI (for a review, see e. g. these *Acta*, **31** (1970), 379—380) and the book under review. The two books complement each other, the former being an elegant introduction which beautifully conveys the spirit of the subject; and the latter a monograph which contains an encyclopaedic discussion of the results known today on bases in Banach spaces as well as in other spaces and of some unsolved problems on them. Some so far unpublished results and observations of the author have also been included in this latter work.

To make the book under review accessible to a larger circle of readers the author gives exact references to textbooks when basic results from functional analysis are used without proof. Results which have appeared only in periodicals are usually proved as lemmas.

Chapter I begins with the basis problem, i. e. whether or not every separable Banach space possesses a basis, and enumerates some of its reformulations. The basis problem was first raised explicitly in the famous book of BANACH, and despite many efforts in solving it, it has remained one of the most significant open problems of functional analysis.

Next, the reader will find a thorough discussion of some of the deeper properties of the most important types of bases for Banach spaces in a great detail. The reviewer should especially like to emphasize the exhaustive presentation of the following two topics. The first concerns properties of strong duality and weak duality, which can be formulated as follows: given a basis $\{x_n\}$ for a Banach space E, what can we say about the sequence $\{f_n\}$ of coefficient functionals associated with $\{x_n\}$ when both E and the conjugate space E^* are endowed with their norm-topologies or weak topologies, respectively. The converse problem is also dealt with. The second one concerns stability theorems of PALEY-WIENER type, in particular the famous KREIN—MILMAN—RUTMAN theorem. These theorems assert that various properties of a sequence $\{x_n\}$ in a Banach space are "stable" in the sense that they are preserved by every sequence $\{x_n\}$ "sufficiently near" the sequence $\{x_n\}$.

Chapter II contains a relatively full account of special classes of bases in Banach spaces. It is divided into two parts: I. Classes of Bases not Involving Unconditional Convergence, and II. Unconditional Bases and Some Classes of Unconditional Bases.

Separate sections are devoted to a systematic study of each of the following particular classes of bases: monotone and strictly monotone bases, normal bases, positive bases, retro-bases in conjugate spaces, shrinking bases and boundedly complete bases, both considered and studied first in detail by R. C. JAMES, Besselian and Hilbertian bases, bases of types P and P^*, bases of types l_+ and $(l_+)^*$, etc; such special classes of unconditional bases as orthogonal and hyperorthogonal bases, which are the "unconditional analogues" of monotone bases, symmetric and subsymmetric bases, perfectly homogeneous bases, etc; and, furthermore, absolutely convergent bases and uniform bases.

One of the main problems that are considered for each special class of bases is whether or not there exists in every separable Banach space a basis belonging to the respective class. In finite dimensional Banach spaces the solutions, in general, are known, and with a few exceptions they are obvious. In infinite dimensional Banach spaces the answer to the corresponding existence problems is either negative or unknown (an affirmative answer would also imply an affirmative answer to the basis problem). In the first case counter examples are given, while in the second case one considers the more restricted problem of the existence of bases of that class in infinite dimensional Banach spaces with bases.

In the course of these investigations, a number of interesting special properties of particular classes of bases are considered, and certain interrelations between these classes are established.

Both chapters end with "Notes and remarks", which have a double purpose. They contain references to original papers in which the principal results in question have been discussed, and in addition, they contain references to some results related to but not included in those given in the text.

The author tried on the whole to adhere to the traditional terminology and notation; there are only a few exceptions. At any rate, a *Notation*, an *Author*, and a *Subject Index* have been provided to give guidance to the reader. Moreover, known interconnections between related concepts and results are sometimes summarized in the form of a table.

The bibliography concerns only the material of Volume I and does not aim at being complete, but wants merely to give a useful orientation to the reader. The bibliography for Volume II will be given separately in that volume.

The book has been carefully and accurately written. The style of its presentation is tight, with hardly a word wasted. There is a disturbing lack of motivation in some places: the author usually states the theorems without background explanations.

To sum up, the present volume contains a great wealth of information in a concise and polished form, and, what is especially welcome, a great number of problems in an explicit form. It will certainly indicate the location of the weak and of the strong spots in the edifice of the theory built so far, and thereby facilitate both the study of the subject, as it exists today, and future research on it.

The second volume, in preparation, will treat upon the following topics: Generalizations of the notion of a basis; Applications to the study of the structure of Banach spaces; Some properties of bases in concrete Banach spaces; Bases in general (not necessarily separable) Banach spaces; Bases in topological linear spaces.

F. Móricz (Szeged)

Lajos Takács, Combinatorial Methods in the Theory of Stochastic Processes (Wiley Series in Probability and Mathematical Statistics), XI+262 pages, New York—London—Sidney, John Wiley & Sons, Inc., 1967.

As the author writes in the Introduction, the aim of this valuable book is to show that for wide classes of random variables and stochastic processes the problem of finding the distribution of the supremum for both sums of random variables and sample functions of stochastic processes can be solved in an elementary way, and this problem, in turn, frequently arises in various fields in the theory of probability. Most of the results presented were achieved by the author during the period 1961—1966 and some have already been published in a series of papers. The book is diveded into eight chapters.

Chapter 1 (Ballot theorems) contains a generalization of the following classical ballot theorem of J. Bertrand. If in a ballot candidate A scores a votes and candidate B scores b votes, $a \geqq b$, then the probability that A is leading throughout the counting of the votes is $(a-b)$ $(a+b)$, provided all the possible voting records are equally probable. TAKÁCS's generalization, which serves as a base for all the subsequent considerations of the book, is the following. *Let $\varphi(u)$ be a nondecreasing function on $[0, t]$ for which $\varphi'(u)=0$ almost everywhere and $\varphi(0)=0$. Let $\varphi(t+u)=\varphi(t)+\varphi(u)$ for $u \in [0, t]$. Define $\delta(u)=1$ if $\varphi(v)-v \leqq \varphi(u)-u$ for every $v \in [u, u+t]$, and $\delta(u)=0$ otherwise. Then $\int_0^t \delta(u)\,du = t-\varphi(t)$ whenever $\varphi(t) \leqq t$.**)

Chapter 2 (Fluctuations of sums of random variables) deals with the determination of the distribution of the maximum of sums of cyclically interchangeable, interchangeable and independent, identically distributed random variables and gives also a discrete generalization of the classical ruin problem.

Chapter 3 (Fluctuations of sample functions of stochastic processes) determines the distribution of the supremum of stochastic processes whose increments are cyclically interchangeable, interchangeable, or stationary independent, and proves a continuous generalization of the classical ruin problem.

Chapter 4 (Random walk processes) treats special processes of the above kind and a random walk process and the Brownian motion process.

Chapter 5 (Queuing processes), Chapter 6 (Dam and storage processes) and Chapter 7 (Risk processes) demonstrate further applications of the preceding general theorems in the theories determined by the chapter headings in the brackets.

Chapter 8 (Order statistics) starts with another extension of the ballot theorem, then by this extension the author gives new proofs and generalizations of the results of Gnedenko, Koroljuk Mihalevič, Smirnov, Birnbaum, Pyke, Tingey and others concerning the exact distribution of Kolmogorov — Smirnov — Rényi type statistics and also computes the exact distribution of the one sided random sample-size M. Kac and similar statistics.

There is an Appendix containing various topics referred to in the text. After every chapter there are some problems (most of them are not merely exercises, but are intended to be extensions of the material covered) whose complete solutions may be found at the end of the book. Each chapter ends with a carefully compiled bibliography and the author should be praised also for commenting historically most of the problems. The good didactical structure and the clear style make the book very well readable.

S. Csörgő (Szeged)

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*) We remark that L. GEHÉR has given a further generalization of Takács's theorem the proof of which is very simple. His paper appeared in these *Acta*, **29** (1968), 163— 165.

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ACTA SCIENTIARUM MATHEMATICARUM

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